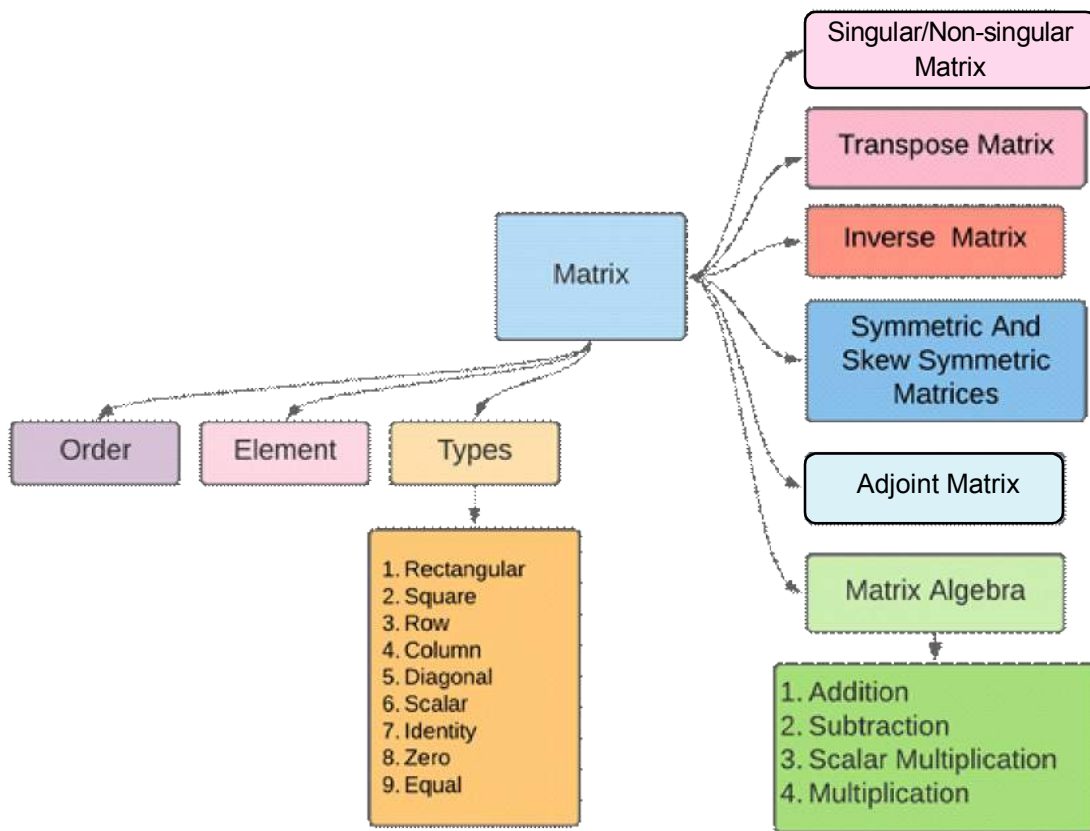


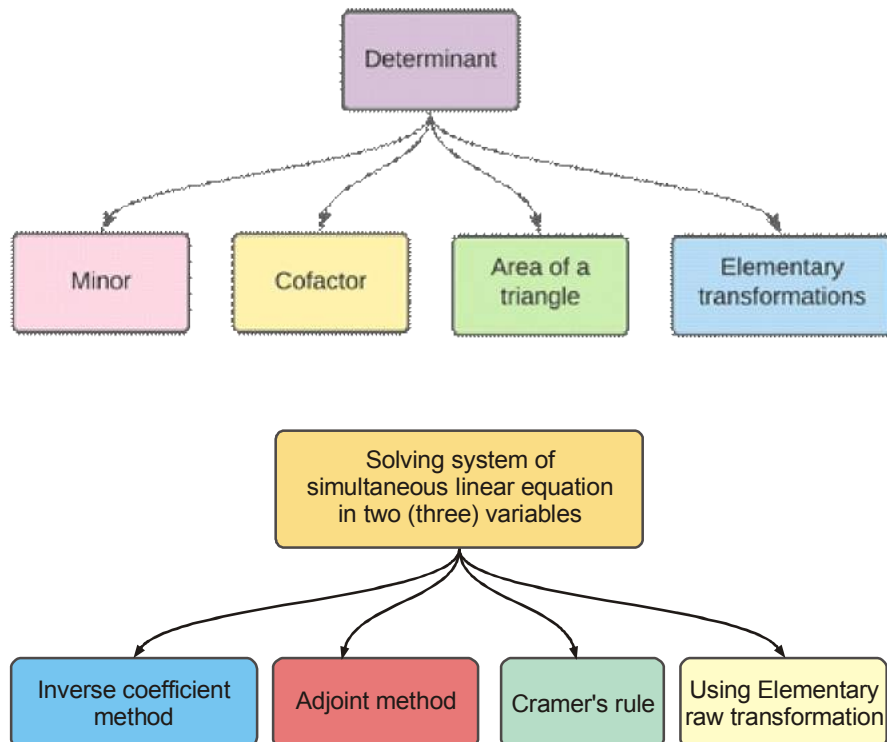
**2.0 LEARNING OUTCOMES**

After completion of the unit the students will be able to:

- ❖ Understand and use concept of a matrix and related terms
- ❖ Differentiate between types of matrices
- ❖ Perform mathematical operations:
  - (i) Addition and subtraction of matrices, properties of addition and subtraction of matrices
  - (ii) Scalar multiplication and Multiplication of matrices, properties of multiplication of matrices
- ❖ Understand and use concept of determinant of a matrix.
- ❖ Understand and use properties of determinant
- ❖ Use elementary operations (transformations) to evaluate determinant value
- ❖ Apply concept of determinant to solve problems based on simple applications such as area of triangle and collinearity of the three points.
- ❖ Understand the conditions for inverse matrix to exist and find inverse of a matrix
- ❖ Solve the system of linear equation using:
  - (i) Cramer's rule
  - (ii) Inverse of coefficient matrix method
  - (iii) Adjugate method
  - (iv) Row reduction method
- ❖ Apply concept of matrix and determinant to formulate and solve real- life situations
- ❖ Able to perform simple applications of matrices and determinants including *Leontief input output model* for two variables



## 2.1 Concept map



## 2.2 Introduction



The Republic day parade is a treat for the eyes of every proud Indian. The order and the synchronisation among the army personnel, moving on the beat of drums is a sight to marvel. You can also see the group move together with each personnel holding on to his position in the arrangement of rows and columns.

Another example of elements in a formation is the position and arrangement of students in a class photograph and how easy



it is to spot a student standing in the first row and the second column from the top left!

Have you ever had difficulty in finding your seat in a cinema hall? Ever wondered why cinema halls and sports arena have seats arranged in the rows and column formations?

Yes! Among many reasons, it is helpful to locate position of each element in the arrangement of things.

## 2.3 Matrix

**Definition:** A rectangular array or table of numbers, symbols, expressions or functions, when arranged in rows and columns, is known as a *matrix* (plural *matrices*). Each member of this arrangement is called an *element* of the matrix.

If we want to write an arrangement of numbers, say 1, 3 and -4

Using matrix, we can write it as  $[1 \quad 3 \quad -4]$  or  $\begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$

Let us now try to arrange x, y, z, w, r and u.

Using matrix, we can write it in so many ways as

$\begin{bmatrix} x & y & z \\ w & r & u \end{bmatrix}$  or  $\begin{bmatrix} x & y \\ z & w \\ r & u \end{bmatrix}$  and many more

A matrix is expressed by using capital English alphabet, say  $A = [a_{ij}]$  where  $a_{ij}$  is the element at the  $i$ -th row and  $j$ -th column of the matrix and  $1 \leq i \leq m, 1 \leq j \leq n, \forall i, j \in \mathbb{N}$ .  $a_{ij}$  is also known as general element of matrix  $A$ .

For example, In matrix  $A = \begin{bmatrix} 2+i & 5 \\ -1 & 2k \\ 4-2i & 0 \end{bmatrix}$

The elements in matrix  $A$  are written as:

$$\begin{aligned}
a_{11} &= 2 + i \\
a_{12} &= 5 \\
a_{21} &= -1, \\
a_{22} &= 2k, \\
a_{31} &= 4 - 2i \text{ and } a_{32} = 9
\end{aligned}$$

and

**Definition:** For a matrix  $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$  having 'm' number of rows and 'n' number of columns, the expression  $m \times n$  is called the *order* of the given matrix A where  $1 \leq i \leq m, 1 \leq j \leq n, \forall i, j \in \mathbb{N}$  and is read as 'm by n'.

The elements  $a_{11}, a_{22}, a_{33}, \dots$  where  $a_{ij}, \forall i = j$  are called elements of the *diagonal* a diagonal elements of the matrix

And the elements where  $a_{ij}, \forall i \neq j$  are called elements of the *non-diagonal* a non-diagonal elements of the matrix

The order of matrix  $A = \begin{bmatrix} 2 & 5 \\ -1 & 2 \\ 4 & 0 \end{bmatrix}$  is  $3 \times 2$ , having 6 elements in total

$B = \begin{bmatrix} x & y & z \\ w & r & u \end{bmatrix}$  is a matrix of order  $2 \times 3$ , having 6 elements

Though the elements of a matrix can take any value, for the scope of this chapter we shall consider only those matrices whose elements are real numbers or functions taking real values

### Example 1

Shalabh has 3 books, 2 pens and 3 notebooks while Rashmi has 1 pen, 4 books and 5 notebooks in their respective school bags. Express the information as a matrix. What is the order of the matrix obtained?

**Solution:** The information can be represented in tabular form as

|         | Books | Pens | Notebooks |
|---------|-------|------|-----------|
| Shalabh | 3     | 2    | 3         |
| Rashmi  | 4     | 1    | 5         |

We can represent the given information as  $A = \begin{bmatrix} 3 & 2 & 3 \\ 4 & 1 & 5 \end{bmatrix}$

Here first column shows the number of books, second column represents the number of pens and third column shows the number of notebooks the students have in their respective bags.

The order of matrix A is  $2 \times 3$ , having 6 elements.

Is that the only way to express the given information?

No, another orientation of the same information can be written as matrix  $B = \begin{bmatrix} 3 & 4 \\ 2 & 1 \\ 3 & 5 \end{bmatrix}$

where first row denotes the number of books, second row denotes the number of pens while the third row denotes the number of notebooks each student has

The number of elements in matrix B is still 6 while the order is  $3 \times 2$

Can you think of any other way to represent the information?

### Example 2

Write the coordinates of triangle ABC with vertices A (4, -1), B (3, 2) and C (2, -4) in a matrix formation

**Solution:** Vertices of triangle ABC can be written in matrix form in two ways,

$$X = \begin{matrix} A \\ B \\ C \end{matrix} \begin{bmatrix} 4 & -1 \\ 3 & 2 \\ 2 & -4 \end{bmatrix} \text{ of order } 3 \times 2 \text{ or } Y = \begin{matrix} A & B & C \\ \begin{bmatrix} 4 & 3 & 2 \\ -1 & 2 & -4 \end{bmatrix} \end{matrix} \text{ of order } 2 \times 3$$

### Example 3

If a matrix has 4 elements, what are the possible orders such a matrix can have?

**Solution:** In a matrix is of order  $m \times n$ , we have  $mn$  number of elements.

therefore, to find all possible orders of a matrix with 4 elements, we will find all ordered pairs of natural numbers, whose product is 4.

Hence, this matrix can have possible orders:  $1 \times 4$ ,  $4 \times 1$ ,  $2 \times 2$

### Example 4

Construct a  $3 \times 3$  matrix whose elements are given by  $a_{ij} = \frac{(i+2j)}{5}$

**Solution:** In general a  $3 \times 3$  matrix is given by  $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\text{For } a_{11}, i = 1 \text{ and } j = 1 \Rightarrow a_{11} = \frac{(1 + 2.1)}{5} = \frac{3}{5}$$

$$\text{Similarly, For } a_{12}, i = 1 \text{ and } j = 2 \Rightarrow a_{12} = \frac{(1 + 2.2)}{5} = 1$$

$$\text{Similarly, } a_{13} = \frac{(1 + 2.3)}{5} = \frac{7}{5}$$

$$a_{21} = \frac{(2 + 2.1)}{5} = \frac{4}{5}$$

$$a_{22} = \frac{(2 + 2.2)}{5} = \frac{6}{5}$$

$$a_{23} = \frac{(2 + 2.3)}{5} = \frac{8}{5}$$

$$a_{31} = \frac{(3 + 2.1)}{5} = 1$$

$$a_{32} = \frac{(3 + 2.2)}{5} = \frac{7}{5}$$

$$a_{33} = \frac{(3 + 2.3)}{5} = \frac{9}{5}$$

Therefore 
$$A = \begin{bmatrix} \frac{3}{5} & 1 & \frac{7}{5} \\ \frac{4}{5} & \frac{6}{5} & \frac{8}{5} \\ 1 & \frac{7}{5} & \frac{9}{5} \end{bmatrix}$$

As observed in all matrices discussed till now, we have seen that there can be different orientations of a matrix. Let us now discuss the different types of matrices.

### 2.3.1 TYPES OF MATRICES

1. **Rectangular matrix:** A matrix in which number of rows is not equal to number of columns is called a *Rectangular matrix*.

For example, matrix  $A = \begin{bmatrix} 2 & 5 \\ -1 & 2 \\ 4 & 0 \end{bmatrix}$  is a rectangular matrix of order  $3 \times 2$

2. **Square Matrix:** A matrix in which the number of rows and the number of columns are equal is called a *Square matrix*.

For example matrices  $B = \begin{bmatrix} 0 & -4 & 3 \\ 1 & 0 & -7 \\ 2 & 2 & 0 \end{bmatrix}$  is a square matrix of order  $3 \times 3$  or B is a square matrix of order 3.

3. **Row Matrix:** A matrix having exactly one row for a given number of columns is called a *Row matrix*. For example, Matrix  $D = [2 \ 3]$  is a row matrix of order  $1 \times 2$

4. **Column Matrix:** A matrix having exactly one column for a given number of rows is called a *Column matrix*

For example, Matrix  $C = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$  is a column matrix of order  $3 \times 1$

5. **Diagonal Matrix:** A square matrix in which all the non-diagonal entries are zero i.e.,  $a_{ij} = 0, \forall i \neq j$  is called a *Diagonal matrix*

For example,  $P = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$  and  $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 4 \end{bmatrix}$  are diagonal matrices of order  $2 \times 2$  and  $3 \times 3$  respectively

6. **Scalar Matrix:** A diagonal matrix having same diagonal elements,

i.e.,  $a_{ij} = k, \forall i = j$  where  $k \neq 0$  is called a *Scalar matrix*.

For example, matrices  $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$  and  $B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$  are scalar matrices of order  $2 \times 2$  and  $3 \times 3$  respectively

7. **Identity Matrix:** A scalar matrix in which all the diagonal entries are equal to 1,

i.e.  $a_{ij} = 1 \forall i = j$  and  $a_{ij} = 0, \forall i \neq j$  is called an *Identity matrix*, denoted by English alphabet  $I$

It is also known as *Unit matrix*. A unit matrix of order  $n$  is written as  $I_n$ .

Identity matrices  $I = [1], \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  are identity matrices of order  $1 \times 1, 2 \times 2$  and  $3 \times 3$  respectively

Note that a scalar matrix with 1 as each diagonal element is an identity matrix

8. **Zero Matrix:** A matrix with each of its elements as zero,

i.e.  $a_{ij} = 0, \forall 1 \leq i \leq m, 1 \leq j \leq n, \forall i, j \in \mathbb{N}$  is called as *zero matrix*

For example, matrix  $[0], \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$  are zero matrices of different orders. A zero matrix is denoted by English alphabet  $O$ , sometimes also called a null or void matrix as well.

9. **Equal Matrices:** Two matrices  $A = [a_{ij}]$  and  $B = [b_{ij}]$  having same order  $m \times n$  are called *Equal matrices* when each element of  $A$  is equal to the corresponding element of  $B$ ,

i.e.  $a_{ij} = b_{ij} \forall 1 \leq i \leq m \text{ and } 1 \leq j \leq n$

In such a case we denote equal matrices as  $A = B$

### Example 5

If  $A = \begin{bmatrix} 1 & 2a \\ -8 & b+1 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & -6 \\ -8 & 13 \end{bmatrix}$  are equal matrices, find the values of  $a$  and  $b$

**Solution:**

As  $A = B \Rightarrow a_{ij} = b_{ij} \forall 1 \leq i \leq m \text{ and } 1 \leq j \leq n (\because A = B)$

Therefore,  $2a = -6 \Rightarrow a = -3$  and  $b + 1 = 13 \Rightarrow b = 12$

### Example 6

Find the values of  $a, b, c$ , and  $d$  from the following equation:

$$\begin{bmatrix} 4 & 24 \\ -3 & 11 \end{bmatrix} = \begin{bmatrix} 2a+b & 4c+3d \\ a-2b & 5c-d \end{bmatrix}$$



**Solution:** As the given matrices are equal,

therefore by equating the corresponding elements of both matrices we get:

$$2a + b = 4$$

$$a - 2b = -3$$

$$4c + 3d = 24$$

$$\text{And } 5c - d = 11$$

Solving these equations, we get  $a = 1$ ,  $b = 2$ ,  $c = 3$  and  $d = 4$

### 2.3.2 CHECK YOUR PROGRESS

#### EXERCISE - A

1. Identify the type of matrices given below and write the order of each matrix:

i)  $A = \begin{bmatrix} 2 & 3 \end{bmatrix}$

ii)  $B = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

iii)  $C = \begin{bmatrix} 2 & 5 \\ -1 & 2 \\ 4 & 0 \end{bmatrix}$

iv)  $D = \begin{bmatrix} 1 & 3 & -4 \end{bmatrix}$

v)  $E = \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix}$

vi)  $P = \begin{bmatrix} 0 & -4 & 3 \\ 1 & 0 & -7 \\ 2 & 2 & 0 \end{bmatrix}$

vii)  $Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

viii)  $R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

ix)  $X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

x)  $Z = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

2. i)  $A = \begin{bmatrix} 0 & -4 & 3 \\ 1 & 0 & -7 \\ 2 & 2 & 0 \end{bmatrix}$ , write the element  $a_{12}$

ii)  $B = \begin{bmatrix} -9 & 4 & -3 \\ -1 & 0 & 4 \\ 2 & 2 & 0 \end{bmatrix}$ , find the sum of elements at  $b_{22}$  and  $b_{32}$

iii)  $C = \begin{bmatrix} -9 & 4 & -3 \\ -1 & 0 & 4 \\ 2 & 2 & 0 \end{bmatrix}$ , find  $c_{21} + c_{32} - c_{13}$

3. Construct matrix  $A = [a_{ij}]$  of order  $2 \times 3$  where  $a_{ij} = \frac{(i+2j)^2}{2}$

4. Construct matrix  $B = [b_{ij}]$  of order  $2 \times 2$  where  $b_{ij} = \frac{|i-j|}{3}$

5. How many distinct  $2 \times 2$  matrices can be formed by using numbers 5, 7 and -1? Justify your answer.



6. A matrix has 14 elements. How many matrices of different orders are possible?
7. Find the values of a, b, c and d from the equation:

$$\begin{bmatrix} 14 & a+b \\ c+d & b+c \end{bmatrix} = \begin{bmatrix} a & -b \\ 8 & 0 \end{bmatrix}$$

## 2.4 ALGEBRA OF MATRICES

Now we are going to learn how to apply basic mathematical operations such as

- Multiplication of a matrix by a scalar value
- Addition and subtraction of two or more matrices

Multiplication of two or more matrices

### 2.4.1 MULTIPLICATION OF A MATRIX BY A SCALAR VALUE

A baker bakes two types of breads using the proportions of ingredients as follows:

This information can be denoted by:

$$A = \begin{matrix} \text{Bread I} & \text{Flour} & \text{Butter} & \text{Milk} \\ \text{Bread II} & \begin{bmatrix} 2 & 50 & 20 \\ 3 & 25 & 10 \end{bmatrix} \end{matrix}$$

On a Sunday, the baker plans to make the breads thrice as much as he does on a regular day. We know that he will have to take thrice the quantity he uses for regular days.

| Items    | Flour | Butter | Milk |
|----------|-------|--------|------|
| Bread I  | 2kg   | 50g    | 20ml |
| Bread II | 3kg   | 25g    | 10ml |

In this case,  $3A = 3 \times A$

$$= 3 \times \begin{matrix} \text{Bread I} & \text{Flour} & \text{Butter} & \text{Milk} \\ \text{Bread II} & \begin{bmatrix} 2 & 50 & 20 \\ 3 & 25 & 10 \end{bmatrix} \end{matrix}$$



*Note that-*

- New matrix is obtained by multiplying each element of matrix A by 3
- Scalar multiplication is irrespective of the order of the given matrix

$$\begin{aligned} &= \begin{matrix} \text{Bread I} & \text{Flour} & \text{Butter} & \text{Milk} \\ \text{Bread II} & \begin{bmatrix} 3 \times 2 & 3 \times 50 & 3 \times 20 \\ 3 \times 3 & 3 \times 25 & 3 \times 10 \end{bmatrix} \end{matrix} \\ &= \begin{matrix} \text{Bread I} & \text{Flour} & \text{Butter} & \text{Milk} \\ \text{Bread II} & \begin{bmatrix} 6 & 150 & 60 \\ 9 & 75 & 30 \end{bmatrix} \end{matrix} \end{aligned}$$

**Definition:** For a matrix  $A = [a_{ij}]$  of order  $m \times n$  and  $k$  is a scalar quantity, then  $kA$  is another matrix obtained by multiplying each element of  $A$  by the scalar quantity  $k$ , i.e.  $kA = k[a_{ij}] = [ka_{ij}]$ ,  $\forall 1 \leq i \leq m$  and  $1 \leq j \leq n$ . The order of the new matrix remains the same as that of the order of the given matrix.

## Properties of scalar Multiplication

If  $A = [a_{ij}]$  and  $B = [b_{ij}]$  are two matrices of the same order, say  $m \times n$ , and  $k$  and  $p$  are scalars, then

$$(i) \quad k(A + B) = kA + kB$$

$$(ii) \quad (k + p)A = kA + pA$$

$$(iii) \quad k(A - B) = kA - kB$$

**Definition:** For a non-zero matrix  $A$ , of order  $m \times n$ , a matrix  $B$  of the same order is called *Negative matrix* of matrix  $A$ . If  $A + B = O$ , where  $O$  is the zero matrix of the same order. We denote negative matrix  $A$  as  $-A$

$$\text{For example, for } A = \begin{bmatrix} 11 & -b \\ -8c & 9 \end{bmatrix},$$

$$\begin{aligned} \text{Negative } A &= -A = (-1) \times \begin{bmatrix} 11 & -b \\ -8c & 9 \end{bmatrix} \\ &= \begin{bmatrix} (-1) \times 11 & (-1) \times -b \\ (-1) \times -8c & (-1) \times 9 \end{bmatrix} \\ &= \begin{bmatrix} -11 & b \\ 8c & -9 \end{bmatrix} \end{aligned}$$

$$\text{Note that } A + (-A) = \begin{bmatrix} 11 & -b \\ -8c & 9 \end{bmatrix} + \begin{bmatrix} -11 & b \\ 8c & -9 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

## 2.4.2 ADDITION OF MATRICES

Let us consider a scenario where students of three sections of class XII are to be divided for two group activities in the math lab. The details of boys and girls for the activities are as given below:

| Class   | Number of boys |          | Number of girls |          |
|---------|----------------|----------|-----------------|----------|
|         | Group I        | Group II | Group I         | Group II |
| XII - A | 6              | 4        | 8               | 6        |
| XII - B | 5              | 5        | 9               | 4        |
| XII - C | 8              | 4        | 7               | 5        |

Table 2.1(i)

$$\text{Let matrix A represent group I students as } G_1 = \begin{matrix} A & \begin{bmatrix} 6 & 8 \\ 5 & 9 \\ 8 & 7 \end{bmatrix} \\ \text{Boys} & \text{Girls} \end{matrix}$$

$$\text{And matrix B, represent group II students as } G_2 = \begin{matrix} B & \begin{bmatrix} 4 & 6 \\ 5 & 4 \\ 4 & 5 \end{bmatrix} \\ \text{Boys} & \text{Girls} \end{matrix}$$

Suppose the teacher in-charge needs a compiled list of total number of boys and girls for both the activities, then we can represent it as an addition of matrices A and B as shown below:

$$\begin{aligned}
 G_1 + G_2 &= \begin{matrix} A \begin{bmatrix} 6 & 8 \end{bmatrix} \\ B \begin{bmatrix} 5 & 9 \end{bmatrix} \\ C \begin{bmatrix} 8 & 7 \end{bmatrix} \end{matrix} + \begin{matrix} A \begin{bmatrix} 4 & 6 \end{bmatrix} \\ B \begin{bmatrix} 5 & 4 \end{bmatrix} \\ C \begin{bmatrix} 4 & 5 \end{bmatrix} \end{matrix} \\
 &\quad \text{Boys Girls Boys Girls} \\
 &= \begin{matrix} A \begin{bmatrix} 6+4 & 8+6 \end{bmatrix} \\ B \begin{bmatrix} 5+5 & 9+4 \end{bmatrix} \\ C \begin{bmatrix} 8+4 & 7+5 \end{bmatrix} \end{matrix} = \begin{matrix} A \begin{bmatrix} 10 & 14 \end{bmatrix} \\ B \begin{bmatrix} 10 & 13 \end{bmatrix} \\ C \begin{bmatrix} 12 & 12 \end{bmatrix} \end{matrix} \\
 &\quad \text{Boys Girls Boys Girls}
 \end{aligned}$$

**Note that-**

- Both the matrices are of equal order
- The sum of matrices is obtained by adding corresponding elements of the matrices

**Definition:** For two matrices  $A = [a_{ij}]$  and  $B = [b_{ij}]$ , of same order  $m \times n$ , the sum of two matrices A and B is defined as a matrix  $S = A+B = [s_{ij}]$  of order  $m \times n$  such that

$$s_{ij} = a_{ij} + b_{ij} \quad \forall 1 \leq i \leq m \text{ and } 1 \leq j \leq n$$

### Properties of addition of matrices

- Addition of two or more matrices is possible only when the given matrices are of the same order. The order of resultant matrix is also same on the order of the given matrices.  
 $\Rightarrow A_{m \times n} + B_{m \times n} = C_{m \times n}$
- Matrix addition is Commutative  $\Rightarrow A_{m \times n} + B_{m \times n} = B_{m \times n} + A_{m \times n}$
- Matrix addition is Associative  $\Rightarrow A_{m \times n} + (B_{m \times n} + C_{m \times n}) = (A_{m \times n} + B_{m \times n}) + C_{m \times n}$
- Zero matrix is the additive identity  $\Rightarrow A_{m \times n} + O_{m \times n} = A_{m \times n} = O_{m \times n} + A_{m \times n}$
- Negative of a matrix is the additive inverse of the given matrix.  
 $\Rightarrow A_{m \times n} + (-A_{m \times n}) = O_{m \times n}$

### 2.4.3 SUBTRACTION OF MATRICES

From the table 2.1(i), if we need to know how many more boys and girls are there in group I as compared to group II

Clearly that would need us to find  $G_1 - G_2$  matrix

**Definition:** For two matrices  $A = [a_{ij}]$  and  $B = [b_{ij}]$ , of same order  $m \times n$ , the difference of the two matrices A and B is defined as a matrix  $D = A - B = [d_{ij}]$  of order  $m \times n$  such that  $d_{ij} = a_{ij} - b_{ij} \quad \forall 1 \leq i \leq m \text{ and } 1 \leq j \leq n$

$$\begin{aligned}
 \text{From table 2.1(i), } G_1 - G_2 &= \begin{matrix} A \begin{bmatrix} 6 & 8 \end{bmatrix} \\ B \begin{bmatrix} 5 & 9 \end{bmatrix} \\ C \begin{bmatrix} 8 & 7 \end{bmatrix} \end{matrix} - \begin{matrix} A \begin{bmatrix} 4 & 6 \end{bmatrix} \\ B \begin{bmatrix} 5 & 4 \end{bmatrix} \\ C \begin{bmatrix} 4 & 5 \end{bmatrix} \end{matrix} \\
 &\quad \text{Boys Girls Boys Girls} \\
 &= \begin{matrix} A \begin{bmatrix} 6-4 & 8-6 \end{bmatrix} \\ B \begin{bmatrix} 5-5 & 9-4 \end{bmatrix} \\ C \begin{bmatrix} 8-4 & 7-5 \end{bmatrix} \end{matrix} = \begin{matrix} A \begin{bmatrix} 2 & 2 \end{bmatrix} \\ B \begin{bmatrix} 0 & 5 \end{bmatrix} \\ C \begin{bmatrix} 4 & 2 \end{bmatrix} \end{matrix} \\
 &\quad \text{Boys Girls Boys Girls}
 \end{aligned}$$

### Example 7

For the matrices  $A = \begin{bmatrix} 3 & 4 & 0 \\ -1 & 12 & 3 \\ 6 & 1 & 2 \end{bmatrix}$ ,  $B = \begin{bmatrix} 7 & 7 & 2 \\ -11 & 0 & 2 \\ 3 & -1 & 4 \end{bmatrix}$  and

$C = \begin{bmatrix} 11 & 2 & -3 \\ 2 & 0 & 1 \\ 1 & 9 & 1 \end{bmatrix}$ , calculate

- i)  $2A$
- ii)  $A + B - C$
- iii)  $3B - A + 2C$

**Solution:**

$$\text{i) } 2A = 2 \times \begin{bmatrix} 3 & 4 & 0 \\ -1 & 12 & 3 \\ 6 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 8 & 0 \\ -2 & 24 & 6 \\ 12 & 2 & 4 \end{bmatrix}$$

$$\begin{aligned} \text{ii) } A + B - C &= \begin{bmatrix} 3 & 4 & 0 \\ -1 & 12 & 3 \\ 6 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 7 & 7 & 2 \\ -11 & 0 & 2 \\ 3 & -1 & 4 \end{bmatrix} - \begin{bmatrix} 11 & 2 & -3 \\ 2 & 0 & 1 \\ 1 & 9 & 1 \end{bmatrix} = \\ &= \begin{bmatrix} 3+7-11 & 4+7-2 & 0+2-(-3) \\ -1+(-11)-2 & 12+0-0 & 3+2-1 \\ 6+3-1 & 1+(-1)-9 & 2+4-1 \end{bmatrix} = \begin{bmatrix} -1 & 9 & 5 \\ -14 & 12 & 4 \\ 8 & -9 & 5 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{iii) } 3B - A + 2C &= 3 \times \begin{bmatrix} 7 & 7 & 2 \\ -11 & 0 & 2 \\ 3 & -1 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 4 & 0 \\ -1 & 12 & 3 \\ 6 & 1 & 2 \end{bmatrix} + 2 \times \begin{bmatrix} 11 & 2 & -3 \\ 2 & 0 & 1 \\ 1 & 9 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 21-3+22 & 21-4+4 & 6-0-6 \\ -33+1+4 & 0-12+0 & 6-3+2 \\ 9-6+2 & -3-1+18 & 12-2+2 \end{bmatrix} \\ &= \begin{bmatrix} 40 & 21 & 0 \\ -28 & -12 & 5 \\ 5 & 14 & 12 \end{bmatrix} \end{aligned}$$

### Example 8

If  $X = \begin{bmatrix} -1 & 3 \\ 8 & 4 \end{bmatrix}$  and  $Y = \begin{bmatrix} -5 & 1 \\ -1 & -2 \end{bmatrix}$  then find the matrix  $Z$ , such that  $2X + Y = 5Z$

**Solution:** As  $2X+Y = 5Z$

Then order of the matrix on LHS and RHS must be same

Let matrix  $Z$  be of order  $2 \times 2$  written as  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\text{Then } 2 \times \begin{bmatrix} -1 & 3 \\ 8 & 4 \end{bmatrix} + \begin{bmatrix} -5 & 1 \\ -1 & -2 \end{bmatrix} = 5 \times \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -7 & 7 \\ 15 & 6 \end{bmatrix} = \begin{bmatrix} 5a & 5b \\ 5c & 5d \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{-7}{5} & \frac{7}{5} \\ 3 & \frac{6}{5} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

### Example 9

Find X and Y if  $X+Y = \begin{bmatrix} -1 & 13 \\ 2 & 4 \end{bmatrix}$  and  $X - Y = \begin{bmatrix} 5 & -8 \\ 3 & 0 \end{bmatrix}$

**Solution:**  $(X+Y) + (X-Y) = \begin{bmatrix} -1 & 13 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 5 & -8 \\ 3 & 0 \end{bmatrix}$

$$\Rightarrow 2X = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix} \Rightarrow X = \begin{bmatrix} 2 & \frac{5}{2} \\ \frac{5}{2} & 2 \end{bmatrix}$$

Also,  $(X+Y) - (X-Y) = \begin{bmatrix} -1 & 13 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 5 & -8 \\ 3 & 0 \end{bmatrix}$

$$\Rightarrow 2Y = \begin{bmatrix} -6 & 21 \\ -1 & 4 \end{bmatrix} \Rightarrow Y = \begin{bmatrix} -3 & \frac{21}{2} \\ -\frac{1}{2} & 2 \end{bmatrix}$$

### 2.4.4 MULTIPLICATION OF MATRICES

Here is another application of matrix algebra in a real-life situation, Neha private limited company hired a public-relations firm to promote their business by distributing free sample of two products. The cost of promotion per sample of each product (in rupees) is given by:

$$C = \begin{array}{c} \text{Product I} \\ \text{Product II} \end{array} \begin{bmatrix} 4.50 \\ 5 \end{bmatrix}$$

*Cost per sample*

The number of products of each type promoted among men and women in a city is given by

matrix  $X = \begin{array}{cc} \text{Men} & \begin{bmatrix} 20000 & 10000 \end{bmatrix} \\ \text{Women} & \begin{bmatrix} 15000 & 21000 \end{bmatrix} \end{array}$

*Product I   Product II*

Using matrix algebra, find the total amount spent by the company for their promotion campaign

The total cost of promoting among men =  $20000 \times 4.50 + 10000 \times 5$

$$= 90000 + 50000 = 140000$$

And, total cost of promoting among women =  $15000 \times 4.50 + 21000 \times 5$

$$= 67500 + 105000 = 172500$$

In this case of multiplication of two matrices X and C, the number of columns in X should be equal to the number of rows in C.

Furthermore, for getting the elements of the product matrix, we take rows of A and columns of B, multiply them element-wise and take the sum as shown below;

Using matrix algebra, we can find  $XC =$

$$\begin{array}{cc} \text{Men} & \begin{bmatrix} 20000 & 10000 \end{bmatrix} \\ \text{Women} & \begin{bmatrix} 15000 & 21000 \end{bmatrix} \end{array} \times \begin{array}{c} \text{Product I} \\ \text{Product II} \end{array} \begin{bmatrix} 4.50 \\ 5 \end{bmatrix}$$

*Product I   Product II                      Cost per contact*

$$\begin{aligned}
 &= \begin{array}{l} \text{Promotion among men} \\ \text{Promotion among women} \end{array} \begin{bmatrix} 20000 \times 4.50 + 10000 \times 5 \\ 15000 \times 4.50 + 21000 \times 5 \end{bmatrix} \\
 &\qquad\qquad\qquad \text{Cost (in rupees)} \\
 &= \begin{bmatrix} 140000 \\ 172500 \end{bmatrix}
 \end{aligned}$$

**Definition:** For two matrices  $A = [a_{ij}]$  of order  $m \times n$  and  $B = [b_{jk}]$  of order  $n \times p$ , the multiplication of the matrices  $AB$  is defined as a matrix  $P = AB = [p_{ik}]$  of order  $m \times p$  such that for finding the element  $p_{ik}$ , we multiple  $i$ th row of first matrix  $A$  with the  $k$ th column of second matrix  $B$  and calculate the sum of these products

$$\text{i.e. } p_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + a_{i3}b_{3k} + a_{i4}b_{4k} + \dots + a_{in}b_{nk} \quad \forall 1 \leq i \leq m \text{ and } 1 \leq k \leq n$$

### Example 10

Let  $A = \begin{bmatrix} 3 & 5 \\ -4 & 6 \end{bmatrix}$  and  $B = \begin{bmatrix} -9 & 2 \\ 1 & -7 \end{bmatrix}$ , find  $AB$  and  $BA$ .

**Solution:** Before finding the product of two matrices, it is important to check if the order of multiplication is well defined.

Since the number of columns in matrix  $A$  is equal to the number of rows in matrix  $B$ , also the number of columns in matrix  $B$  is equal to the number of rows in matrix  $A$ , products  $AB$  and  $BA$  are defined and each will be of order  $2 \times 2$ .

$$\begin{aligned}
 \text{Hence } AB &= \begin{bmatrix} 3 & 5 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} -9 & 2 \\ 1 & -7 \end{bmatrix} \\
 &= \begin{bmatrix} (3)(-9) + (5)(1) & (3)(2) + (5)(-7) \\ (-4)(-9) + (6)(1) & (-4)(2) + (6)(-7) \end{bmatrix} = \begin{bmatrix} -22 & -29 \\ 42 & -50 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 BA &= \begin{bmatrix} -9 & 2 \\ 1 & -7 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ -4 & 6 \end{bmatrix} \\
 &= \begin{bmatrix} (-9)(3) + (2)(-4) & (-9)(5) + (2)(6) \\ (1)(3) + (-7)(-4) & (1)(5) + (-7)(6) \end{bmatrix} \\
 &= \begin{bmatrix} -35 & -33 \\ 31 & -37 \end{bmatrix}
 \end{aligned}$$

If  $A, B$  are matrices of orders respectively  $m \times n, k \times l$ , then both  $AB$  and  $BA$  are defined if and only if  $n = k$  and  $l = m$ .

Observe that in this case  $AB \neq BA$

This observation does not mean that defined multiplication of matrices is not commutative

For example, let  $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$  and  $B = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}$

Clearly,  $AB$  and  $BA$  are defined and  $AB = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} = BA$

• Multiplication of diagonal matrices is commutative

Recall that for two real numbers 'x' and 'y' when  $xy = 0$  then either  $x = 0$  or  $y = 0$ .

Let us see if is true in the case of multiplication of matrices as well



Consider two non-zero matrices  $A = \begin{bmatrix} 0 & -9 \\ 0 & 4 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & -5 \\ 0 & 0 \end{bmatrix}$

$$\text{Here } AB = \begin{bmatrix} 0 & -9 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Whereas neither of the two matrices A or B are zero matrices. Thus, if the product of two matrices is a zero matrix, then it is not necessary that one of the matrices is a zero matrix

### Example 11

$$\text{Find } \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 9 & 3 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

$$\begin{aligned} \text{Solution: } \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 9 & 3 & -3 \\ 6 & 1 & -2 \end{bmatrix} &= \begin{bmatrix} -2 - 9 + 12 & 0 - 3 + 2 & 1 + 3 - 4 \\ 0 + 18 - 18 & 0 + 6 - 3 & 0 - 6 + 6 \\ -6 - 18 + 24 & 0 - 6 + 4 & 3 + 6 - 8 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 0 \\ 0 & -2 & 0 \end{bmatrix} \end{aligned}$$

### Properties of multiplication of matrices

- The multiplication of matrices is associative i.e. for any three matrices A, B and C,  $(AB)C = A(BC)$ , whenever order of multiplication is defined on both sides.
- Distributive property of multiplication holds true for multiplication of matrices. i.e. For three matrices A, B and C  
 $A(B+C) = AB + AC$   
and,  $A(B - C) = AB - AC$ , whenever order of multiplication is defined on both sides.
- For a given square matrix A of order  $m \times m$ , there exists a multiplicative identity matrix **I**, of the same order such that  $IA = AI = A$ .

### Example 12

For  $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ , prove that  $A^2 - 4A + I = O$ , where O is a zero matrix.

**Solution:** For matrix A of order  $2 \times 2$ , we will take **I** identity matrix of the same order

$$\begin{aligned} \text{Now } A^2 = A.A &= \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A^2 - 4A + I &= \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - 4 \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 7 - 8 + 1 & 12 - 12 + 0 \\ 4 - 4 + 0 & 7 - 8 + 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O \end{aligned}$$



### Example 13

Given that  $X_{2 \times n}$ ,  $Y_{3 \times k}$ ,  $Z_{2 \times p}$ ,  $W_{n \times 3}$  and  $P_{p \times k}$  are matrices of specified order. What are the conditions

- for  $n$ ,  $k$  and  $p$  so that  $3PY + 2WY$  is defined
- for the order of the matrix  $2X - 3Z$

**Solution:**

- For defined multiplication of matrices  $P$  and  $Y$ , number of columns of matrix  $P$  must be equal to number of rows of matrix  $Y \Rightarrow k = 3$

Then Order of  $PY = p \times k$

Also order of  $WY = n \times k$

Then for  $3PY + 2WY$  to be defined the order of  $PY$  and  $WY$  must be same

$$\Rightarrow p \times k = n \times k \Rightarrow p = n$$

- $2X - 3Z$  is defined if order  $X$  is equal to order of  $Z \Rightarrow 2 \times n = 2 \times p \Rightarrow p = n$

## 2.5 SPECIAL MATRICES

### 2.5.1 TRANSPOSE OF A MATRIX

**Definition:** For a matrix  $A = [a_{ij}]$  of order  $m \times n$ , the matrix obtained by interchanging the rows and columns of the matrix  $A$  is called the transpose of matrix  $A$ .

$$\text{If } A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} \text{ then } A' = \begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix}$$

$$B = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 3 & -3 \\ 6 & 1 & -2 \end{bmatrix} \text{ then } B' = \begin{bmatrix} -2 & 9 & 6 \\ 0 & 3 & 1 \\ 1 & -3 & -2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -2 & 3 \\ 3 & 2 & -1 \end{bmatrix} \text{ then } P' = \begin{bmatrix} 1 & 3 \\ -2 & 2 \\ 3 & -1 \end{bmatrix}$$

Transpose of a matrix  $A$  is denoted by  $A'$  or  $A^T$ .

### Example 14

$$\text{If } A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 & 3 \\ 5 & -4 \end{bmatrix} \text{ then find the matrix } A' - 2B$$

$$\text{Solution: } A' - 2B = \begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix} - 2 \begin{bmatrix} -2 & 3 \\ 5 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 2+4 & 1-6 \\ 3-10 & -4+8 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 \\ -7 & 4 \end{bmatrix}$$

For a matrix  $A$  of order  $m \times n$ , the order of transpose matrix  $A'$  is  $n \times m$ .

## Properties of Transpose of a matrix

For given matrices A and B:

- $(A')' = A$
- $(kA)' = kA'$  (where k is any constant)
- $(A + B)' = A' + B'$
- $(A B)' = B' A'$

### 2.5.2 SYMMETRIC AND SKEW SYMMETRIC MATRICES

**Definition:** For a given square matrix  $A = [a_{ij}]$ , if  $A' = A$ ,  $\forall 1 \leq i \leq m$  and  $1 \leq j \leq n$ , then the matrix A is called a **symmetric matrix**.

For example,  $P = \begin{bmatrix} \sqrt{3} & -8 \\ -8 & 9 \end{bmatrix}$  is a symmetric matrix as  $P' = P$

$A = \begin{bmatrix} 2 & 2 & 3 \\ 2 & -1 & -2 \\ 3 & -2 & 1 \end{bmatrix}$  is a symmetric matrix as  $A' = A$

**Definition:** For a given square matrix  $A = [a_{ij}]$ ,

if  $A' = -A$ ,  $\forall 1 \leq i \leq m$  and  $1 \leq j \leq n$ , then the matrix A is called a **skew-symmetric matrix**.

All diagonal elements of a skew-symmetric matrix are zero

For example,  $B = \begin{bmatrix} 0 & -8 \\ 8 & 0 \end{bmatrix}$  and  $B' = \begin{bmatrix} 0 & 8 \\ -8 & 0 \end{bmatrix} = -B$

$\Rightarrow$  is a skew-symmetric matrix.

For  $Q = \begin{bmatrix} 0 & -2 & -8 \\ 2 & 0 & 4 \\ 8 & -4 & 0 \end{bmatrix}$ ,  $Q' = \begin{bmatrix} 0 & 2 & 8 \\ -2 & 0 & -4 \\ -8 & 4 & 0 \end{bmatrix} = -Q$

$\Rightarrow Q$  is a skew-symmetric matrix

For a square matrix A having real values as elements,

- $A + A'$  is a symmetric matrix
- $A - A'$  is a skew symmetric matrix.

#### Example 15

If  $P = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$ , then verify that  $P'P = I$ , where I is an identity matrix

**Solution:**  $P = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \Rightarrow P' = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$   
 $\Rightarrow P'P = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$   
 $= \begin{bmatrix} \cos^2 x + \sin^2 x & -\cos x \sin x + \cos x \sin x \\ -\cos x \sin x + \cos x \sin x & \cos^2 x + \sin^2 x \end{bmatrix}$   
 $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

### Example 16

If A, B are symmetric matrices of same order, then what can be said for matrix  $AB - BA$  ?

**Solution:** As A, B are symmetric matrices

$$\Rightarrow A' = A \text{ and } B' = B \text{ —————(i)}$$

$$\text{Now } (AB - BA)' = (AB)' - (BA)'$$

$$= (B'A') - (A'B')$$

$$= BA - AB \quad [\text{from (i)}]$$

$$= -(AB - BA)$$

$$\Rightarrow AB - BA \text{ is a skew-symmetric matrix}$$

### Example 17

Express the matrix  $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & -1 & 0 \\ 3 & 5 & 1 \end{bmatrix}$  as the sum of a symmetric and a skew-symmetric matrix.

**Solution:**  $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & -1 & 0 \\ 3 & 5 & 1 \end{bmatrix}$

$$A' = \begin{bmatrix} 1 & -4 & 3 \\ 2 & -1 & 5 \\ 3 & 0 & 1 \end{bmatrix}$$

Let  $P = \frac{1}{2} (A + A') = \frac{1}{2} \left[ \begin{bmatrix} 1 & 2 & 3 \\ -4 & -1 & 0 \\ 3 & 5 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -4 & 3 \\ 2 & -1 & 5 \\ 3 & 0 & 1 \end{bmatrix} \right] = \frac{1}{2} \begin{bmatrix} 2 & -2 & 6 \\ -2 & -2 & 5 \\ 6 & 5 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1 & -1 & 3 \\ -1 & -1 & \frac{5}{2} \\ 3 & \frac{5}{2} & 1 \end{bmatrix}$$

Also  $P' = \begin{bmatrix} 1 & -1 & 3 \\ -1 & -1 & \frac{5}{2} \\ 3 & \frac{5}{2} & 1 \end{bmatrix} = P$

$\Rightarrow P$  is a symmetric matrix

Let  $Q = \frac{1}{2} (A - A') = \frac{1}{2} \left[ \begin{bmatrix} 1 & 2 & 3 \\ -4 & -1 & 0 \\ 3 & 5 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -4 & 3 \\ 2 & -1 & 5 \\ 3 & 0 & 1 \end{bmatrix} \right] = \frac{1}{2} \begin{bmatrix} 0 & 6 & 0 \\ -6 & 0 & -5 \\ 0 & 5 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 3 & 0 \\ -3 & 0 & -\frac{5}{2} \\ 0 & \frac{5}{2} & 0 \end{bmatrix}$$

$$\text{Also } Q' = = \begin{bmatrix} 0 & -3 & 0 \\ 3 & 0 & \frac{5}{2} \\ 0 & \frac{-5}{2} & 0 \end{bmatrix} = -Q$$

$\Rightarrow Q$  is a skew – symmetric matrix.

$$\text{Also } P + Q = \begin{bmatrix} 1 & -1 & 3 \\ -1 & -1 & \frac{5}{2} \\ 3 & \frac{5}{2} & 1 \end{bmatrix} + \begin{bmatrix} 0 & 3 & 0 \\ -3 & 0 & \frac{-5}{2} \\ 0 & \frac{5}{2} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ -4 & -1 & 0 \\ 3 & 5 & 1 \end{bmatrix} = A$$

A is represented as the sum of a symmetric and a skew- symmetric matrix.

### 2.5.3 CHECK YOUR PROGRESS

#### EXERCISE–B

- Complete the following table.

Order of the matrix

| A            | B            | $A \pm B$    | AB           |
|--------------|--------------|--------------|--------------|
| $2 \times 2$ | $2 \times 2$ |              |              |
| $2 \times 3$ | $3 \times 2$ |              |              |
| $3 \times 4$ | $4 \times 1$ |              |              |
| $3 \times 3$ | $3 \times 3$ |              |              |
| $2 \times 3$ |              | $2 \times 3$ |              |
|              | $3 \times 2$ |              | $1 \times 2$ |
| $2 \times 3$ |              | $2 \times 3$ |              |
| $1 \times 3$ | $3 \times 2$ |              |              |

- For  $A = \begin{bmatrix} 6 & -5 \\ -7 & 4 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$  and  $C = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}$ , show that
  - Commutative property does not hold true for multiplication of matrices A and B i.e.  $AB \neq BA$
  - Associative property holds true for multiplication of three matrices, i.e.  $A(BC) = (AB)C$
- Consider  $A = \begin{bmatrix} 1 & 3 & 4 \\ -2 & 1 & 2 \\ 3 & -2 & 1 \end{bmatrix}$  verify the  $A.I = I.A = A$ , where I is the identity matrix of order  $3 \times 3$
- If  $A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$  and  $B = \begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$  then show that
  - $(A + B)' = (A)' + (B)'$
  - $(AB)' = (B)'(A)'$

5. Do as directed

i) For  $A = \begin{bmatrix} 6 & -5 \\ -7 & 4 \end{bmatrix}$ , find  $A^2 - 6A$

ii) Evaluate  $\begin{bmatrix} 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

iii) Find a matrix A such that  $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix} = \begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} - A$

iv) If the matrix  $X = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$  is equal to the matrix  $Y = \begin{bmatrix} p - q & 2p + r \\ 2p - q & 3r + s \end{bmatrix}$  then find value of p, q, r and s.

v) Let  $P = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$  and  $Q = \begin{bmatrix} 1 & 4 \\ 7 & 2 \end{bmatrix}$  then calculate  $3P - 2Q$ .

6. If  $A = \begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}$  and  $B = \begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix}$ , then find a matrix C, such that  $3A - 2B + 4C = 0$

7. Given  $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ ,  $B = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$ , find:

- $2A - 3B$
- $AB$
- $BA$
- $AB - BA$

8. For  $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$  show that  $A^3 - 23A - 40I = O$ , where I is an identity matrix of order 3, and

O is zero matrix

9. Two booksellers A and B sell the textbook of Mathematics and Applied Mathematics. In the month of march, bookseller A sold 250 books of Mathematics and 400 books of Applied Mathematics whereas bookseller B sold 230 books of Mathematics and 425 books of Applied Mathematics. In the month of April, bookseller A sold 550 books of Mathematics and 300 books of Applied Mathematics and bookseller B sold 270 books of Mathematics and 450 books of Applied Mathematics.

Represent the given information into matrix form and find the total sale for both the booksellers in the month of March and April, using matrix algebra.

10. Cost of a pen and a notebook are Rs.12 and Rs. 27 respectively. On a given day, shopkeeper P sells 5 pens and 7 notebooks, whereas shopkeeper Q sells 6 pens and 4 notebooks on a particular day. Find the income of both the shopkeepers, using matrix algebra

## 2.6 DETERMINANT

Recall that for a given system of linear equations

$$\text{i.e. } a_1 x + b_1 y = c_1$$

$$\text{and, } a_2 x + b_2 y = c_2$$

the system of equations has a unique solution if  $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \Rightarrow a_1 b_2 \neq a_2 b_1$

$$\Rightarrow a_1 b_2 - a_2 b_1 = 0$$

In this chapter we have learnt that.  $a_1 x + b_1 y = c_1$

and,  $a_2 x + b_2 y = c_2$  can be represented in matrix form as:

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

The expression  $a_1 b_2 - a_2 b_1$  is associated with the square matrix  $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$  and is called determinant of the matrix and denoted as  $\det A$  or  $|A|$  and sometimes by Greek symbol  $\Delta$ .

Every square matrix is associated with a number  $C$  real or complex, known as Determinant of the matrix  $A$  and is denoted by  $\det A$  or  $|A|$ , or  $A$  determinant is also expressed using greek symbol  $\Delta$ .

**Definition:** For a given square matrix  $A = [a_{ij}]$ , of order  $m$ , a number (real or complex) is known as determinant of the matrix  $A$

For matrix  $A = [a]$   
of order  $1 \times 1$   
 $\det A = a$

Let  $X$  be a set of square matrices and  $R$  be the set of numbers (real or complex) such that a function  $f$  is defined as  $f : M \rightarrow K$  by  $f(A) = k$ , where

$A \in X$  and  $k \in R$ , then  $f(A)$  is called the determinant of  $A$ .

Only a square matrix has a determinant value associated to it

For example, For  $A = [2]$ , a matrix of order  $1 \times 1$

$$\det A = |A| = 2$$

### 2.6.1 MINOR OF AN ELEMENT

**Definition:** In a given determinant, minor of an element  $a_{ij}$  is the determinant obtained by deleting its  $i$ th row and  $j$ th column in which element  $a_{ij}$  lies, and denoted by  $M_{ij}$ .

#### Example 18:

Find the minor of element -11 in  $A = \begin{bmatrix} 6 & -11 \\ -1 & 3 \end{bmatrix}$

**Solution:** As -11 is  $(1, 2)^{\text{th}}$  element of the matrix

Minor of an element of a determinant of order  $m$  ( $m \geq 2$ ) is a determinant of order  $n - 1$

We will eliminate first row and second column  $\begin{bmatrix} 6 & -11 \\ -1 & 3 \end{bmatrix}$  and  $M_{12} = -1$

For a matrix  $A = \begin{bmatrix} 6 & -5 \\ -7 & 4 \end{bmatrix}$  of order  $2 \times 2$ ,

**determinant of A** =  $\det A = |A|$  is represented as  $\begin{vmatrix} 6 & -5 \\ -7 & 4 \end{vmatrix}$

And calculated as  $|A| = \begin{vmatrix} 6 & -5 \\ -7 & 4 \end{vmatrix}$

$$= [6 \times 4] - [(-5) \times (-7)]$$

$$= 24 - 35$$

$$= -11.$$

### Example 19

Given  $A = \begin{bmatrix} 3 & 5 \\ -4 & 6 \end{bmatrix}$  and  $B = \begin{bmatrix} -9 & 2 \\ 1 & -7 \end{bmatrix}$ , find the following

- i)  $|A|$       ii)  $|B|$       iii)  $2|A|$       iv)  $|2A|$       v)  $|A| |B|$       vi)  $|AB|$

**Solution:**

i)  $|A| = \begin{vmatrix} 3 & 5 \\ -4 & 6 \end{vmatrix} = (3)(6) - (-4)(5) = 18 + 20 = 38$

ii)  $|B| = \begin{vmatrix} -9 & 2 \\ 1 & -7 \end{vmatrix} = (-9)(-7) - (1)(2) = 63 - 2 = 61$

iii)  $2|A| = 2 \begin{vmatrix} 3 & 5 \\ -4 & 6 \end{vmatrix} = 2(38) = 76$

iv)  $A \begin{bmatrix} 3 & 5 \\ -4 & 6 \end{bmatrix}$ ,  $2A = 2 \begin{bmatrix} 3 & 5 \\ -4 & 6 \end{bmatrix} = \begin{bmatrix} 6 & 10 \\ -8 & 12 \end{bmatrix}$

$$|2A| = \begin{vmatrix} 6 & 10 \\ -8 & 12 \end{vmatrix} = (6)(12) - (-8)(10) = 72 + 80 = 152$$

v)  $|A| |B| = \begin{vmatrix} 3 & 5 \\ -4 & 6 \end{vmatrix} \begin{vmatrix} -9 & 2 \\ 1 & -7 \end{vmatrix} = 38 \cdot 61 = 2318$

vi)  $AB = \begin{bmatrix} -22 & -29 \\ 42 & -50 \end{bmatrix}$ ,

$$|AB| = \begin{vmatrix} -22 & -29 \\ 42 & -50 \end{vmatrix} = (-22)(-50) - (42)(-29) = 1100 + 1218 = 2318$$

### Example 20

Find  $x$  if  $\begin{vmatrix} 3 & -6 \\ 4 & 0 \end{vmatrix} = \begin{vmatrix} 3 & x^2 \\ x & -1 \end{vmatrix}$

**Solution:** As  $\begin{vmatrix} 3 & -6 \\ 4 & 0 \end{vmatrix} = \begin{vmatrix} 3 & x^2 \\ x & -1 \end{vmatrix}$

$$\Rightarrow 0 + 24 = -3 - x^3$$

$$\Rightarrow x^3 = -27$$

$$\therefore x = -3$$



### Example 21

Find the minor of element 7 in the determinant  $B = \begin{vmatrix} 2 & 3 & 0 \\ -3 & 1 & 7 \\ 1 & -2 & 5 \end{vmatrix}$

**Solution:** As 7 is  $(2,3)^{\text{th}}$  element of the determinant  $\begin{vmatrix} 2 & 3 & 0 \\ -3 & 1 & 7 \\ 1 & -2 & 5 \end{vmatrix}$

We shall eliminate second row and the third column and  $M_{23} = \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = 2 \times (-2) - 3 \times 1 = -4 - 3 = -7$

### 2.6.2 COFACTOR OF AN ELEMENT OF A DETERMINANT

**Definition:** In a square matrix, cofactor of an element  $a_{ij}$ , denoted by  $A_{ij}$  or  $C_{ij}$  is defined by  $A_{ij} = (-1)^{i+j} M_{ij}$ , where  $M_{ij}$  is the minor of  $a_{ij}$

In example 17,  $A_{12} = (-1)^{1+2} M_{12} = (-1)(-4) = 4$

In example 21,  $A_{23} = (-1)^{2+3} M_{23} = (-1)(-7) = 7$

To evaluate a determinant, elements of a row or a column are multiplied with their corresponding cofactors.

For a determinant of order  $3 \times 3$ , the value of the determinant can be evaluated by expanding along the first row as  $a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$ .

Now let us consider another square matrix  $B = [b_{ij}]$ , of order  $3 \times 3$

$$\text{let } B = \begin{bmatrix} 2 & -1 & -3 \\ -3 & 0 & 5 \\ 1 & -1 & -2 \end{bmatrix}$$

For a matrix  $B$  of order  $3 \times 3$ ,  $\det B$  is calculated in terms of determinants of order  $2 \times 2$ . We can expand and calculate  $\det B$  in six ways by expanding with respect to any one of the three rows ( $R_1, R_2$  and  $R_3$ ) or any one of the three columns ( $C_1, C_2$  and  $C_3$ ) giving the same determinant value

**Here, we shall multiply the elements of any one row or column by their respective cofactors. For example, if we use  $R_1$  to find  $\det B$ , it will be as follows:**

$$\begin{aligned} \det B &= \begin{vmatrix} 2 & -1 & -3 \\ -3 & 0 & 5 \\ 1 & -1 & -2 \end{vmatrix} = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13} \\ &= 2 \begin{vmatrix} 0 & 5 \\ -1 & -2 \end{vmatrix} + 1 \begin{vmatrix} -3 & 5 \\ 1 & -2 \end{vmatrix} - 3 \begin{vmatrix} -3 & 0 \\ 1 & -1 \end{vmatrix} \\ &= 2[0 - (-5)] + 1[6 - 5] - 3[3 - 0] \\ &= 10 + 1 - 9 = 2 \end{aligned}$$

If the determinant of any matrix is zero then that matrix is called Singular Matrix.

### Example 22

Evaluate  $\Delta = \begin{vmatrix} 0 & \cos x & -\sin x \\ -\cos x & 0 & \cos y \\ \sin x & -\cos y & 0 \end{vmatrix}$

**Solution:** Expanding along  $R_1$

$$\Delta = 0 - \cos x (0 - \sin x \cos y) - \sin x (\cos x \cos y - 0) = \sin x \cos x \cos y - \sin x \cos x \cos y = 0$$

### Example 23

Find minors and cofactors of the elements of the determinant

$A = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$  and check whether  $a_{11} A_{31} + a_{12} A_{32} + a_{13} A_{33} = 0$

When elements of a row or a column of a determinant are multiplied with cofactors of any other row or column, then their sum is always zero

**Solution:**  $M_{11} = \begin{vmatrix} 0 & 4 \\ 5 & -7 \end{vmatrix} = 0 - 20 = -20$  and  $A_{11} = (-1)^{1+1} M_{11} = -20$

$$M_{12} = \begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix} = -42 - 4 = -46 \text{ and } A_{12} = (-1)^{1+2} M_{12} = 46$$

$$M_{13} = \begin{vmatrix} 6 & 0 \\ 1 & 5 \end{vmatrix} = 30 - 0 = 30 \text{ and } A_{13} = (-1)^{1+3} M_{13} = 30$$

$$M_{21} = \begin{vmatrix} -3 & 5 \\ 5 & -7 \end{vmatrix} = 21 - 25 = -4 \text{ and } A_{21} = (-1)^{2+1} M_{21} = 4$$

$$M_{22} = \begin{vmatrix} 2 & 5 \\ 1 & -7 \end{vmatrix} = -14 - 5 = -19 \text{ and } A_{22} = (-1)^{2+2} M_{22} = -19$$

$$M_{23} = \begin{vmatrix} 2 & -3 \\ 1 & 5 \end{vmatrix} = 10 + 3 = 13 \text{ and } A_{23} = (-1)^{2+3} M_{23} = -13$$

$$M_{31} = \begin{vmatrix} -3 & 5 \\ 0 & 4 \end{vmatrix} = -12 - 0 = -12 \text{ and } A_{31} = (-1)^{3+1} M_{31} = -12$$

$$M_{32} = \begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix} = 8 - 30 = -22 \text{ and } A_{32} = (-1)^{3+2} M_{32} = 22$$

$$M_{33} = \begin{vmatrix} 2 & -3 \\ 6 & 0 \end{vmatrix} = 0 + 18 = 18 \text{ and } A_{33} = (-1)^{3+3} M_{33} = 18$$

$$\text{Now } a_{11} A_{31} + a_{12} A_{32} + a_{13} A_{33} = 2(-12) - 3 \times 22 + 5 \times 18 = 0$$

### 2.6.3 ADJOINT OF A MATRIX

**Definition:** The transpose of cofactors of a matrix of a square matrix  $A = [a_{ij}]$  is called Adjoint matrix of matrix  $A$  and is denoted by  $\text{adj } A$ .

Using the cofactor values of corresponding elements of given matrix  $A$  in example above, we can

write  $\text{adj } A = \begin{bmatrix} -20 & 46 & 30 \\ 4 & -19 & -13 \\ -12 & 22 & -18 \end{bmatrix}' = \begin{bmatrix} -20 & 4 & -12 \\ 46 & -19 & 22 \\ 30 & -13 & -18 \end{bmatrix}$

### Properties of adjoint of a matrix

For a given square matrix  $A$ , of order  $n$ ,

- $A(\text{adj } A) = (\text{adj } A) A = |A| I$ , where  $I$  is the identity matrix of order  $n$
- $|\text{adj}(A)| = |A|^{n-1}$

### Example 24

Find  $k$ , if  $A = \begin{bmatrix} -2 & 3 \\ k & 4 \end{bmatrix}$  is a singular matrix

**Solution:** As  $A$  is a singular matrix  $\Rightarrow |A| = 0$

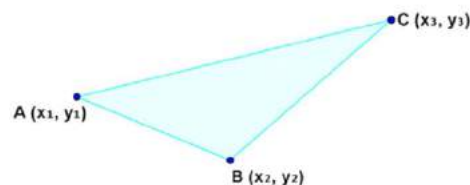
$$\Rightarrow -8 - 3k = 0$$

$$\Rightarrow k = \frac{-8}{3}$$

### 2.6.4 AREA OF TRIANGLE

Let  $A(x_1, y_1)$ ,  $B(x_2, y_2)$  and  $C(x_3, y_3)$  be the vertices of the triangle  $ABC$ , then

$$\text{Area } \Delta ABC = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$



- Since area is a positive quantity, we always take the absolute value of the determinant
- If area is given, use both positive and negative values of the determinant for calculation.
- For given vertices with coordinates  $A(x_1, y_1)$ ,  $B(x_2, y_2)$  and  $C(x_3, y_3)$ ; if the area of a triangle is zero the points  $A$ ,  $B$  and  $C$  are collinear

### Example 25

Find the area of the triangle with vertices  $A(5, 4)$ ,  $B(2, -6)$  and  $C(-2, 4)$ .

**Solution:**  $\text{Area } \Delta ABC = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$$= \frac{1}{2} \begin{vmatrix} 5 & 4 & 1 \\ 2 & -6 & 1 \\ -2 & 4 & 1 \end{vmatrix}$$
$$= \frac{1}{2}(-70) = -35$$

Hence area of the triangle  $= |-35| = 35$  sq. units

### Example 26

For what value of  $k$ , points  $P(3, -2)$ ,  $Q(8, 8)$  and  $R(k, 2)$  are collinear.

**Solution:** As the points  $P$ ,  $Q$  and  $R$  are collinear

$$\Rightarrow \text{Area of triangle PQR} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 3 & -2 & 1 \\ 8 & 8 & 1 \\ k & 2 & 1 \end{vmatrix} = 0$$

Solving above determinant, we get,  $|3(8-2) + 2(8-k) + 1(16-8k)| = 0$

$$|18 + 16 - 2k + 16 - 8k| = 0$$

$$\Rightarrow |-10k + 50| = 0$$

$$\Rightarrow \pm(-10k + 50) = 0 \Rightarrow k = 5$$

### 2.6.5 PROPERTIES OF A DETERMINANT

- (i) For any two square matrices  $A$  and  $B$ , multiplication of their determinant values is same as determinant of their product

For example, let us consider two matrices  $A = \begin{bmatrix} 3 & 5 \\ -4 & 6 \end{bmatrix}$  and  $B = \begin{bmatrix} -9 & 2 \\ 1 & -7 \end{bmatrix}$

$$|A| \cdot |B| = \begin{vmatrix} 3 & 5 \\ -4 & 6 \end{vmatrix} \begin{vmatrix} -9 & 2 \\ 1 & -7 \end{vmatrix} = 38 \times 61 = 2318$$

$$\text{And, } AB = \begin{bmatrix} 3 & 5 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} -9 & 2 \\ 1 & -7 \end{bmatrix} = \begin{bmatrix} -22 & -29 \\ 42 & -50 \end{bmatrix}$$

$$\Rightarrow |AB| = \begin{vmatrix} -22 & -29 \\ 42 & -50 \end{vmatrix} = (-22) \times (-50) - (42) \times (-29) = 1100 + 1218 = 2318$$

Therefore, we can say that for square matrices  $A$  and  $B$ ,  $|A| \times |B| = |AB|$

- (ii) If two square matrices  $A$  and  $B$  of order  $n$  are such that  $A = kB$ , then

$$|A| = k^n |B|, \text{ where } n = 1, 2, 3, \dots$$

Let us consider two matrices  $A = \begin{bmatrix} -4 & 12 \\ 20 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} -1 & 3 \\ 5 & 0 \end{bmatrix}$  of order  $2 \times 2$

Clearly  $A = 4B$

$$\text{And, } |A| = 0 - 240 = -240$$

$$|B| = 0 - 15 = -15$$

$$\Rightarrow |A| = 16 |B| = 4^2 |B|, \text{ where } k = 4 \text{ and } n = 2$$

- iii) The value of the determinant remains unchanged if its rows and columns are interchanged

$$\text{Consider } \Delta = \begin{vmatrix} 5 & -3 & 1 \\ 0 & 2 & -3 \\ 2 & 0 & -1 \end{vmatrix} = 4$$

And  $\Delta' = \begin{vmatrix} 5 & 0 & 2 \\ -3 & 2 & 0 \\ 1 & -3 & -1 \end{vmatrix} = 4$

Therefore, we can say that for a square matrix A;  $\det A = \det A'$  where  $A'$  is the transpose of matrix A

- iv) If any two rows (or columns) of a determinant are interchanged, then sign of determinant changes

Consider  $|A| = \begin{vmatrix} 1 & 3 & 5 \\ -2 & -4 & -2 \\ 4 & 2 & 4 \end{vmatrix} = 42$

If we interchange  $R_1$  with  $R_3$ , then  $\begin{vmatrix} 4 & 2 & 4 \\ -2 & -4 & -2 \\ 1 & 3 & 5 \end{vmatrix} = -42 = -|A|$

The interchange of two rows, say  $i^{th}$  and  $j^{th}$  is denoted as  $R_i \leftrightarrow R_j$  and similarly interchange of  $i^{th}$  and  $j^{th}$  columns is denoted as  $C_i \leftrightarrow C_j$

- v) If two rows  $R_i$  and  $R_j$  (or  $C_i$  and  $C_j$ ) of a determinant are identical, then the value of determinant is zero.

For example  $\Delta = \begin{vmatrix} 5 & -3 & 1 \\ 1 & 2 & -3 \\ 1 & 2 & -3 \end{vmatrix}$

Here  $R_2 = R_3 \Rightarrow \Delta = 0$

- vi) If corresponding elements of any two rows  $R_i$  and  $R_j$  (or columns  $C_i$  and  $C_j$ ) of a determinant are in the same ratio, then the determinant value is zero.

For example,  $\Delta = \begin{vmatrix} -3 & 4 & -6 \\ 11 & 2 & 22 \\ 2 & 1 & 4 \end{vmatrix} = 0$  as  $\frac{C_1}{C_2} = \frac{1}{2}$

- vii) If all the elements in any one row  $R_i$  (or a column  $C_j$ ) of the determinant are zero, then the determinant value is zero.

For example  $\Delta_1 = \begin{vmatrix} 2 & 0 & 5 \\ 6 & 0 & 4 \\ 1 & 0 & -7 \end{vmatrix} = 0$  and  $\Delta_2 = \begin{vmatrix} 0 & 0 \\ 4 & -9 \end{vmatrix} = 0$

- viii) If some or all elements of a row or column of a determinant are expressed as sum of two (or more) terms, then the determinant can be expressed as sum of two or more determinants

For example, to evaluate  $\Delta = \begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix}$

We will split  $R_2$  as follows:

$$\Delta = \begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} + \begin{vmatrix} a & b & c \\ 2x & 2y & 2z \\ x & y & z \end{vmatrix}$$

$$\text{Let } \Delta_1 = \begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} \quad \text{and} \quad \Delta_2 = \begin{vmatrix} a & b & c \\ 2x & 2y & 2z \\ x & y & z \end{vmatrix}$$

As  $R_1 = R_2$  in  $\Delta_1$  and  $R_2 = 2R_1$  in  $\Delta_2$

$$\text{Therefore, } \Delta_1 = \begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} = 0 \quad \text{and} \quad \Delta_2 = \begin{vmatrix} a & b & c \\ 2x & 2y & 2z \\ x & y & z \end{vmatrix} = 0. \quad \text{Hence } \Delta = 0$$

- ix) If each element of a row (or a column) of a determinant is multiplied by a non-zero constant  $k$ , then the determinant value gets multiplied by  $k$

$$\text{For example, For } A = \begin{bmatrix} 3 & 5 \\ -4 & 6 \end{bmatrix}$$

$$|A| = 18 - 20 = -2$$

If we multiple  $R_1$  by 2, we get

$$\begin{vmatrix} 6 & 10 \\ -4 & 6 \end{vmatrix} = 36 - 40 = -4 = 2|A|$$

Therefore, we can say when we apply  $R_i \rightarrow kR_i$  (or  $C_i \rightarrow kC_i$ ) to a determinant  $\Delta$  we get  $k\Delta$

What would happen if we multiply all rows (or columns) of  $|A|$  by 4, we get

$$|4A| = \begin{vmatrix} 12 & 20 \\ -16 & 24 \end{vmatrix} = 288 - 320 = -32 = 4^2 \cdot (-2) = 4^2 |A|$$

$\Rightarrow$  For any non-zero scalar  $k$ ,  $k |A| = k^n |A|$ , where  $n$  is the order of the determinant

- x) If a row  $R_i$  (or a column  $C_i$ ) is added to equimultiples of corresponding elements of another row  $R_j$  (or a column  $C_j$ ), then the value of determinant remains the same.

$$\text{For example, If } \Delta = \begin{vmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix} = 2(1-8) - 7(1-10) + 1(8-10) = -14 + 63 - 2 = 47$$

Then applying operation  $R_1 \rightarrow R_1 + 2R_2$  gives

$$\Delta' = \begin{vmatrix} 4 & 9 & 3 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix} = 4(1-8) - 9(1-10) + 3(8-10) = -28 + 81 - 6 = 47$$

Or applying operation  $C_2 \rightarrow C_2 - C_3$  gives

$$\Delta'' = \begin{vmatrix} 2 & 6 & 1 \\ 1 & 0 & 1 \\ 10 & 7 & 1 \end{vmatrix} = 2(0-7) - 6(1-10) + 1(7-0) = -14 + 54 + 7 = 47$$

Therefore, the value of the determinant remains same if we apply the operation  $R_i \rightarrow R_i + kR_j$  (or  $C_i \rightarrow C_i + kC_j$ ) for a nonzero scalar  $k$

If more than one operation like  $R_i + kR_j$  (or  $C_i \rightarrow C_i + kC_j$ ) is applied in one step, it should be seen that the row (or column) that is affected in one operation should not be used in another operation

### Example 27

Evaluate  $\Delta = \begin{vmatrix} 42 & 2 & 5 \\ 79 & 7 & 9 \\ 29 & 5 & 3 \end{vmatrix}$

**Solution:** Applying  $C_1 \rightarrow C_1 - 8C_3$

Then  $\Delta = \begin{vmatrix} 2 & 2 & 5 \\ 7 & 7 & 9 \\ 5 & 5 & 3 \end{vmatrix}$

$\Rightarrow \Delta = 0$  as  $C_1 = C_2$

### Example 28

Evaluate  $\Delta = \begin{vmatrix} 1 & a & b+c \\ 1 & b & a+c \\ 1 & c & a+b \end{vmatrix}$

**Solution:** Applying  $C_3 \rightarrow C_2 + C_3$

Then  $\Delta = \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix}$

$\Rightarrow \Delta = 0$  as  $\frac{C_2}{C_3} = \frac{1}{a+b+c}$

### Example 29

Given that  $a, b$  and  $c$  are in A.P., evaluate

$$\Delta = \begin{vmatrix} 2y+4 & 5y+7 & 8y+a \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix}$$

**Solution:** As  $a, b$  and  $c$  are in A.P.  $\Rightarrow 2b = a + c$  ————— (i)

Now Applying  $R_1 \rightarrow R_1 + R_3 - 2R_2$

$$\Delta = \begin{vmatrix} 0 & 0 & a+c-2b \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix} \quad \text{————— from (i)}$$

$= 0$



### Example 30

Without expanding prove that

$$\Delta = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz(x-y)(y-z)(z-x)$$

**Solution:** Taking  $x$  common from  $C_1$  and  $y$  common from  $C_2$  and  $z$  common from  $C_3$ , we get

$$\Delta = xyz \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

Applying  $C_1 \rightarrow C_1 - C_2$  and  $C_2 \rightarrow C_2 - C_3$  we get

$$\Delta = xyz \begin{vmatrix} 0 & 0 & 1 \\ x-y & y-z & z \\ x^2-y^2 & y^2-z^2 & z^2 \end{vmatrix}$$

Taking  $(x-y)$  common from  $C_1$  and  $(y-z)$  common from  $C_2$ , we get

$$\Delta = xyz(x-y)(y-z) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & z \\ x+y & y+z & z^2 \end{vmatrix}$$

Applying  $C_2 \rightarrow C_2 - C_1$ , we get

$$\Delta = xyz(x-y)(y-z) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & z \\ x+y & z-x & z^2 \end{vmatrix}$$

Taking  $(z-x)$  common from  $C_2$ , we get

$$\Delta = xyz(x-y)(y-z)(z-x) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & z \\ x+y & 1 & z^2 \end{vmatrix}$$

Expanding  $C_2$ , we get

$$\begin{aligned} \Delta &= xyz(x-y)(y-z)(z-x)[0+0-1(0-1)] \\ &= xyz(x-y)(y-z)(z-x) \end{aligned}$$

### Example 31

Without expanding, evaluate

$$\Delta = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$$

**Solution:** Applying  $C_1 \rightarrow C_1 - C_3$  and  $C_2 \rightarrow C_2 - C_3$  we get

$$\Delta = \begin{vmatrix} (b+c)^2 - a^2 & 0 & a^2 \\ 0 & (c+a)^2 - b^2 & b^2 \\ c^2 - (a+b)^2 & c^2 - (a+b)^2 & (a+b)^2 \end{vmatrix}$$

Taking  $a + b + c$  common from  $C_1$  and  $C_2$ , we get

$$\Delta = (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix}$$

Applying  $R_3 \rightarrow R_3 - R_1 - R_2$

$$\Delta = (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix}$$

Taking 2 common from  $R_3$ , we get

$$\Delta = 2(a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -b & -a & ab \end{vmatrix}$$

Expanding  $C_1$ , we get

$$\begin{aligned} \Delta &= 2b(a+b+c)^2 [(b+c-a)(abc + a^2b - ab^2 + ab^2) + 0 + a^2(0 + bc + ab - b^2)] \\ &= 2abc(a+b+c)^3 \end{aligned}$$

### Example 32

Evaluate without expanding

$$\Delta = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc \left( 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) = abc + ab + bc + ac$$

**Solution:** Apply  $R_1 \rightarrow \frac{R_1}{a}$ ,  $R_2 \rightarrow \frac{R_2}{b}$  and  $R_3 \rightarrow \frac{R_3}{c}$ , we get

$$\Delta = abc \begin{vmatrix} 1 + \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ \frac{1}{a} & 1 + \frac{1}{b} & \frac{1}{c} \\ \frac{1}{a} & \frac{1}{b} & 1 + \frac{1}{c} \end{vmatrix}$$

Applying  $C_1 \rightarrow C_1 + C_2 + C_3$ , we get

$$\Delta = abc \begin{vmatrix} 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} & \frac{1}{b} & \frac{1}{c} \\ 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} & 1 + \frac{1}{b} & \frac{1}{c} \\ 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} & \frac{1}{b} & 1 + \frac{1}{c} \end{vmatrix}$$

Taking  $1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$  common from  $C_1$ , we get

$$\Delta = abc \left( 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \begin{vmatrix} 1 & \frac{1}{b} & \frac{1}{c} \\ 1 & 1 + \frac{1}{b} & \frac{1}{c} \\ 1 & \frac{1}{b} & 1 + \frac{1}{c} \end{vmatrix}$$

Applying  $R_1 \rightarrow R_1 - R_3$ , we get

$$\Delta = abc \left( 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \begin{vmatrix} 0 & 0 & -1 \\ 1 & 1 + \frac{1}{b} & \frac{1}{c} \\ 1 & \frac{1}{b} & 1 + \frac{1}{c} \end{vmatrix}$$

Expanding  $R_1$ , we get

$$\begin{aligned} \Delta &= abc \left( 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \left[ 0 + 0 - 1 \left( \frac{1}{b} - 1 - \frac{1}{b} \right) \right] = abc \left( 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \\ &= abc + ab + bc + ac \end{aligned}$$

## 2.6.6 CHECK YOUR PROGRESS

### EXERCISE-C

1. Evaluate the following

i)  $\begin{vmatrix} 4 & -2 \\ 6 & -3 \end{vmatrix}$

iv)  $\begin{vmatrix} -3 & -1 & 2 \\ 5 & 7 & -8 \\ -2 & -6 & 6 \end{vmatrix}$

ii)  $\begin{vmatrix} 7 & 1 \\ 4 & -7 \end{vmatrix}$

v)  $\begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{vmatrix}$

iii)  $\begin{vmatrix} 3 & 1 \\ -1 & 8 \end{vmatrix}$

- Find the area of the triangle with vertices  $(-2, -3)$ ,  $(-1, -8)$  and  $(3, 2)$ .
- For what value of "k" the points  $(k, 7)$ ,  $(-4, 5)$  and  $(1, -5)$  are collinear.
- Represent the given matrices as the sum of a symmetric and skew symmetric matrices

i.  $\begin{bmatrix} 3 & 1 \\ -1 & 8 \end{bmatrix}$

ii.  $\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$

iii.  $\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$

5. Evaluate using properties of determinants:

$$i. \begin{vmatrix} b-c & c-a & a-b \\ c-a & a-b & b-c \\ a-b & b-c & c-a \end{vmatrix}$$

$$ii. \begin{vmatrix} x+y & y+z & x+z \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$$

$$iii. \begin{vmatrix} x+y & y+z & x+z \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$$

$$iv. \begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$$

$$v. \begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix}$$

6. Prove the following using properties of determinants:

$$i. \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

$$ii. \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix} = xy$$

$$iii. \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

$$iv. \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix} = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma)$$

$$v. \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix} = 1$$

7. Find adjoint A if

$$i. A = \begin{bmatrix} 2 & 1 \\ -3 & 5 \end{bmatrix}$$

$$ii. A = \begin{bmatrix} -52 & 11 \\ 0 & 51 \end{bmatrix}$$

$$iii. A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

## 2.7 INVERSE OF A MATRIX

$$\text{Consider } A = \begin{bmatrix} 2 & 1 \\ -3 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} \frac{5}{13} & \frac{-1}{13} \\ \frac{3}{13} & \frac{2}{13} \end{bmatrix}$$

$$\text{Here, } AB = \begin{bmatrix} 2 & 1 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} \frac{5}{13} & \frac{-1}{13} \\ \frac{3}{13} & \frac{2}{13} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

**Definition:** For a given square matrix  $A$  of order  $n$ , if there exists another square matrix  $B$  of the same order  $n$ , such that  $AB = BA = I$ , then  $A$  is said to be invertible and  $B$  is called the inverse matrix of  $A$  and denoted by  $A^{-1}$

In the example above,  $B = A^{-1} = \begin{bmatrix} 5 & -1 \\ 13 & 13 \\ 3 & 2 \\ 13 & 13 \end{bmatrix}$

### Properties of inverse of a matrix:

- Inverse of a matrix, if it exists, is unique
- For two invertible matrices of same order, say  $A$  and  $B$ , then  $(AB)^{-1} = B^{-1} A^{-1}$
- For an invertible matrix  $A$ ,  $(A^{-1})^{-1} = A$
- For an invertible matrix  $A$ ,  $(A^T)^{-1} = (A^{-1})^T$

There are many methods by which we can find inverse of a given square matrix.

- Elementary operation (transformation) method
- Matrix method/ Adjugate or Adjoint method
- Row reduction method

#### 2.7.1 FINDING INVERSE MATRIX BY ELEMENTARY OPERATIONS (TRANSFORMATION)

We can apply certain operations (transformations) on a matrix on rows or columns, which are known as elementary operations or transformations of a matrix

- Any two rows (or columns) can be interchanged and denoted by  $R_i \leftrightarrow R_j$  for an interchange between  $i^{th}$  and  $j^{th}$  rows ( or  $C_i \leftrightarrow C_j$  for interchange of  $i^{th}$  and  $j^{th}$  columns)
- A row (or column) can be multiplied by a non-zero scalar  $k$ , and denoted by  $R_i \rightarrow kR_i$  (or  $C_i \rightarrow kC_i$ )
- A row (or column) can be added to the equi-multiples of corresponding elements of any other row or column and denoted by  $R_i \rightarrow R_i + kR_j$  (or  $C_i \rightarrow C_i + kC_j$ ) for a non-zero scalar  $k$ .

**For an invertible square matrix  $A$  (i.e.,  $A^{-1}$  exists)**

- Write  $A = IA$  to apply a sequence of row operations till we get,  $I = BA$ , where  $I$  is identity matrix of same order.
- Similarly, we write  $A = AI$  and apply a sequence of column operations till we get,  $I = AB$ , where  $I$  is identity matrix of same order.

In both the cases, the matrix  $B$  is the inverse of  $A$ .

- In case, after applying one or more elementary row (column) operations on  $A = IA$  ( or  $A = AI$ ), if we obtain all zeros in one or more rows of the matrix  $A$  on L.H.S. of the equation, then  $A^{-1}$  does not exist, or  $A$  is not invertible

### Example 33

By using elementary row transformations, find inverse of matrix  $A = \begin{bmatrix} 1 & 0 \\ -3 & 5 \end{bmatrix}$

**Solution:** To apply row transformations, we will consider:

$$A = IA$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

applying  $R_2 \rightarrow R_2 + 3R_1$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} A$$

Applying  $R_2 \rightarrow \frac{R_2}{5}$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{3}{5} & \frac{1}{5} \end{bmatrix} A$$

$$\text{As } I = BA, \text{ where } B = \begin{bmatrix} 1 & 0 \\ \frac{3}{5} & \frac{1}{5} \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{3}{5} & \frac{1}{5} \end{bmatrix}$$

Alternatively, we can also apply column transformations:

$$A = AI$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ -3 & 5 \end{bmatrix} = A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

applying  $C_2 \rightarrow \frac{C_2}{5}$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{bmatrix}$$

Applying  $C_1 \rightarrow C_1 + \frac{3}{5}C_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 \\ \frac{3}{5} & \frac{1}{5} \end{bmatrix}$$

$$\text{As } I = AB, \text{ where } B = \begin{bmatrix} 1 & 0 \\ \frac{3}{5} & \frac{1}{5} \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{3}{5} & \frac{1}{5} \end{bmatrix}$$

### Example 34

Find the inverse of matrix A using elementary row operations where  $A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$

**Solution:** Using row transformations:  $A = IA$

$$\Rightarrow \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying  $R_2 \rightarrow R_2 - 2R_1$  and  $R_3 \rightarrow R_3 - 3R_1$ , we get

$$\Rightarrow \begin{bmatrix} 2 & 3 & 10 \\ 0 & -12 & -15 \\ 0 & 0 & -50 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}_A$$

Applying  $R_1 \rightarrow \frac{R_1}{2}$  and  $R_2 \rightarrow \frac{R_2}{-12}$ , we get

$$\Rightarrow \begin{bmatrix} 1 & \frac{3}{2} & 5 \\ 0 & 1 & \frac{5}{4} \\ 0 & 0 & -50 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{6} & \frac{-1}{12} & 0 \\ -3 & 0 & 1 \end{bmatrix}_A$$

Applying  $R_1 \rightarrow R_1 - \frac{3}{2}R_2$ , we get

$$\Rightarrow \begin{bmatrix} 1 & 0 & \frac{25}{8} \\ 0 & 1 & \frac{5}{4} \\ 0 & 0 & -50 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{8} & 0 \\ \frac{1}{6} & \frac{-1}{12} & 0 \\ -3 & 0 & 1 \end{bmatrix}_A$$

Applying  $R_3 \rightarrow \frac{R_3}{-50}$ , we get

$$\Rightarrow \begin{bmatrix} 1 & 0 & \frac{25}{8} \\ 0 & 1 & \frac{5}{4} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{8} & 0 \\ \frac{1}{6} & -\frac{1}{12} & 0 \\ \frac{3}{50} & 0 & \frac{-1}{50} \end{bmatrix}_A$$

Applying  $R_1 \rightarrow R_1 - \frac{25}{8}R_3$  and  $R_2 \rightarrow R_2 - \frac{5}{4}R_3$ , we get

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{120} & \frac{-1}{12} & \frac{1}{40} \\ \frac{3}{50} & 0 & \frac{-1}{50} \end{bmatrix}_A$$

$$\text{As } I = AB, \text{ where } B = \begin{bmatrix} \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{120} & \frac{-1}{12} & \frac{1}{40} \\ \frac{3}{50} & 0 & \frac{-1}{50} \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{120} & \frac{-1}{12} & \frac{1}{40} \\ \frac{3}{50} & 0 & \frac{-1}{50} \end{bmatrix}$$

### Example 35

Find  $A^{-1}$ , if  $A = \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$

**Solution:** Using column transformations:

$$A = AI$$

$$\Rightarrow \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



Applying  $C_2 \rightarrow C_2 + \frac{1}{5}C_1$ , we get

$$\Rightarrow \begin{bmatrix} 10 & 0 \\ -5 & 0 \end{bmatrix} = A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

On LHS, the elements of second column are all zeroes, therefore,  $A^{-1}$  does not exist

### 2.7.2 FINDING INVERSE MATRIX BY INVERSE OF COEFFICIENT MATRIX METHOD

Recall that for a given square matrix  $A$  of order  $n$ , we have

$A (\text{adj } A) = (\text{adj } A) A = |A| I$ , where  $I$  is the identity matrix of order  $n$

$$\Rightarrow A^{-1} = \frac{1}{|A|}(\text{adj } A),$$

Clearly  $|A| \neq 0 \Rightarrow$  only a non-singular square matrix is invertible

Now let us learn how to use matrix method to find inverse of a non-singular matrix

i.e., we can find inverse of a matrix  $A$  only when  $|A| \neq 0$

recall that the determinant value of a singular matrix is always zero

#### Example 36

Find the inverse of matrix  $A$  using inverse of the coefficient matrix method when

$$A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$$

Solution:  $A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$

Expanding a row or a column, we will first find  $|A| = \begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix} = 1200 \neq 0$

$\Rightarrow A^{-1}$  exists

$$\text{adj } A = \text{transpose of cofactor matrix} = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

$$\Rightarrow \text{inverse of } A = A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{1200} \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{120} & \frac{-1}{12} & \frac{1}{40} \\ \frac{3}{50} & 0 & \frac{-1}{50} \end{bmatrix}$$

## 2.8 SOLVING SYSTEM OF SIMULTANEOUS LINEAR EQUATIONS

### 2.8.1 INVERSE OF COEFFICIENT MATRIX

This method is used to solve a system of equations in two or three variables.

Let us understand how to find inverse of a given square matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  of order  $2 \times 2$

$$\text{Then } A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

#### Example 37

Solve the following system of equation finding the inverse of coefficient matrix:

$$x - y = 5; 2x + 3y = -1$$

Solution: Let us convert given equation in matrix form  $AX=B$ .

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

Here  $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$  is called coefficient matrix

And  $B = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$  is called the constant matrix

Let us find inverse of coefficient matrix  $= A^{-1} = \frac{1}{1 \times 3 - (-1) \times 2} \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$

Now  $AX = B \Rightarrow X = A^{-1}B$

$$\Rightarrow X = \frac{1}{5} \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 15 - 1 \\ -10 - 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{14}{5} \\ \frac{-11}{5} \end{bmatrix}$$

Therefore  $x = \frac{14}{5}$  and  $y = \frac{-11}{5}$

Recall representing a system of linear equation in matrix formation:

#### Example 38

Solve for  $x$ ,  $y$  and  $z$ :

$$2x - 3y = 5; \quad 5x + 3y = 2$$

Solution: Let us write the given system of equations in matrix form

$$\begin{bmatrix} 2 & -3 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Let  $A = \begin{bmatrix} 2 & -3 \\ 5 & 3 \end{bmatrix}$ ,  $X = \begin{bmatrix} x \\ y \end{bmatrix}$  and  $B = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$  so that  $AX = B$

In this representation, matrix  $A$  is called coefficient matrix and  $B$  is called constant matrix.

Now **pre-multiplying both sides by  $A^{-1}$** , we get

$$A^{-1}(AX) = A^{-1} B$$

$$\Rightarrow I X = A^{-1} B$$

$$\Rightarrow X = A^{-1} B$$

$$\text{For } A = \begin{bmatrix} 2 & -3 \\ 5 & 3 \end{bmatrix}$$

$$\text{As } |A| = 6 + 15 = 21 \neq 0 \Rightarrow A^{-1} \text{ exists and } A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

$$\text{Here } \text{adj } A = \begin{bmatrix} 3 & 3 \\ -5 & 2 \end{bmatrix}$$

$$\text{Therefore } A^{-1} = \frac{1}{21} \begin{bmatrix} 3 & 3 \\ -5 & 2 \end{bmatrix}$$

$$\text{As } \begin{bmatrix} 2 & -3 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{21} \begin{bmatrix} 3 & 3 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \frac{1}{21} \begin{bmatrix} 15 + 6 \\ -25 + 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{21} \begin{bmatrix} 21 \\ -21 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Hence  $x = 1$  and  $y = -1$

### Example 39

Solve the following system of equation using matrix method

$$x + y + z = 10, 2x + y = 13, x + y - 4z = 0$$

**Solution:** The given equation in matrix form can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 13 \\ 0 \end{bmatrix}$$

$$\text{where } A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \end{bmatrix}$$

$$\text{As } |A| = 5 \neq 0 \Rightarrow A^{-1} \text{ exists}$$

$$\text{Adj } A = \begin{bmatrix} -4 & 5 & -1 \\ 8 & -5 & 2 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\text{Therefore } A^{-1} = \frac{1}{|A|} (\text{adj}A) = \frac{1}{5} \begin{bmatrix} -4 & 5 & -1 \\ 8 & -5 & 2 \\ 1 & 0 & -1 \end{bmatrix}$$

Hence the matrix equation  $AX=B$  implies

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -4 & 5 & -1 \\ 8 & -5 & 2 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 10 \\ 13 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 25 \\ 15 \\ 10 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$$

Therefore  $x = 5$ ,  $y = 3$  and  $z = 2$

### 2.8.2 CRAMER'S RULE

Steps to Explain Cramer's rule. Cramer's rule is another method to find the solution of system of linear equations.

#### Example 40

Solve the following system of equation using Cramer's rule

$$2x - 3y = 5; 5x + 3y = 2$$

**Solution:** Let us convert given equation in matrix form  $AX=B$ .

$$\begin{bmatrix} 2 & -3 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Here matrix  $A = \begin{bmatrix} 2 & -3 \\ 5 & 3 \end{bmatrix}$  is called coefficient matrix

and  $B = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$  is called constant matrix.

Firstly, we find the determinant of coefficient matrix called as delta ( $\Delta$ ).

$$\text{Here } \Delta = |A| = \begin{vmatrix} 2 & -3 \\ 5 & 3 \end{vmatrix} = 6 - (-15) = 21 \neq 0$$

Now create a new determinant  $\Delta_x$  by replacing "x" column by the elements of the constant matrix B

$$\Rightarrow \Delta_x = \begin{vmatrix} 5 & -3 \\ 2 & 3 \end{vmatrix} = 15 + 6 = 21$$

Similarly, create another determinant  $\Delta_y$  by replacing "y" column

$$\Delta_y = \begin{vmatrix} 2 & 5 \\ 5 & 2 \end{vmatrix} = 4 - 25 = -21$$

For a system of equations  
 $a_1 x + b_1 y = 0$   
 and,  $a_2 x + b_2 y = 0$   
 as the elements of constant matrix are zero, i.e.  $B =$   
 Such system of equations are called *homogenous* equations

Finally  $x$  is obtained by dividing  $\Delta x$  by  $\Delta$ , and  $\Delta y$  by  $\Delta$

$$\Rightarrow x = \frac{\Delta x}{\Delta} = \frac{21}{21} = 1$$

$$\text{And } y = \frac{\Delta y}{\Delta} = -\frac{21}{21} = -1$$

Therefore  $x = 1$  and  $y = -1$

In Cramer's rule, when  $\Delta \neq 0$ , then the given system of equations

- is consistent
- has a unique solution.

### Example 41

Solve the following system of equations using Cramer's rule

$$x + y + z = 10, 2x + y = 13, x + y - 4z = 0$$

**Solution:** Convert the given equations in matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 13 \\ 0 \end{bmatrix}$$

$$\text{Where } A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 10 \\ 13 \\ 0 \end{bmatrix}$$

$$\text{Here } \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \end{vmatrix} = 1[-4-0] - 1[-8-0] + 1[2-1] = 5 \neq 0$$

$$\text{Now } \Delta x = \begin{vmatrix} 10 & 1 & 1 \\ 13 & 1 & 0 \\ 0 & 1 & -4 \end{vmatrix} = 10[-4] - 1[-52] + 1[13] = 25$$

$$\Delta y = \begin{vmatrix} 1 & 10 & 1 \\ 2 & 13 & 0 \\ 1 & 0 & -4 \end{vmatrix} = 15$$

$$\text{And, } \Delta z = \begin{vmatrix} 1 & 1 & 10 \\ 2 & 1 & 13 \\ 1 & 1 & 0 \end{vmatrix} = 10$$

$$\Rightarrow x = \frac{\Delta x}{\Delta} = \frac{25}{5} = 5,$$

$$\Rightarrow y = \frac{\Delta y}{\Delta} = \frac{15}{5} = 3$$

$$\text{and } z = \frac{\Delta z}{\Delta} = \frac{10}{5} = 2$$

Therefore,  $x = 5$ ,  $y = 3$  and  $z = 2$

### Example 42

Solve the following system of equations using Cramer's rule,

$$2x - 3y = 5; -4x + 6y = -10$$

**Solution:** Let us convert given equations in matrix form  $AX=B$ .

$$\begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$$

$$\text{Here } \Delta = \begin{vmatrix} 2 & -3 \\ -4 & 6 \end{vmatrix} = 0$$

$$\Delta x = \begin{vmatrix} 5 & -3 \\ -10 & 6 \end{vmatrix} = 0$$

$$\Delta y = \begin{vmatrix} 2 & 5 \\ -4 & -10 \end{vmatrix} = 0$$

The given system of equation has infinitely many solutions.

In Cramer's rule, when  $\Delta = 0$  and all other delta values  $\Delta x$ ,  $\Delta y$  and  $\Delta z$  are zero; then the system of equations

- Is dependable consistent.
- has infinitely many solutions.

### Example 43

Solve the following system of equations using Cramer's rule

$$x - 2y + 3z = 1, 2x + y - z = 3 \text{ and } 3x - y + 2z = -2$$

**Solution:** Convert given equations in matrix form

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & -1 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & -1 \\ 3 & -1 & 2 \end{vmatrix} = 1[2 \cdot 1] + 2[4 + 3] + 3[-2 \cdot 3] = 0$$

$$\Delta x = \begin{vmatrix} 1 & -2 & 3 \\ 3 & 1 & -1 \\ -2 & -1 & 2 \end{vmatrix} = 1[2 \cdot 1] + 2[6 - 2] + 3[-3 + 2] = 6 \neq 0$$

$$\Delta y = \begin{vmatrix} 1 & 1 & 3 \\ 2 & 3 & -1 \\ 3 & -2 & 2 \end{vmatrix} = 1(6 - 2) - 1(4 + 3) + 3(-4 - 9) = 4 - 7 - 39 = -42 \neq 0$$

$$\Delta z = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & 3 \\ 3 & -1 & -2 \end{vmatrix} = 1(-2 + 3) + 2(-4 - 9) + 1(-2 - 3) = 1 - 26 - 5 = -30 \neq 0$$

Here  $\Delta = 0$  and the other deltas  $\Delta x$ ,  $\Delta y$  and  $\Delta z$  are non-zero values. In such a case the given system of equations is inconsistent and has no solution.

In Cramer's rule, when  $\Delta = 0$  and any one or more of the deltas  $\Delta x$ ,  $\Delta y$  or  $\Delta z$  are non-zero then the given system of equations

- is inconsistent
- has no solution.

### 2.8.3 ROW REDUCTION METHOD

#### Example 44

Solve using row reduction method:  $x - 3y = 9$ ,  $2x + y + 3 = 0$

**Solution:** Write all the given equations in the form  $ax + by = c$

$$\Rightarrow x - 3y = 9$$

$$2x + y = -3$$

Now we shall write the given system of equations in the augmented matrix as shown below:

$$\begin{array}{lcl} \text{Equation (i)} & \longrightarrow & \begin{bmatrix} 1 & -3 & 9 \end{bmatrix} \\ \text{Equation (ii)} & \longrightarrow & \begin{bmatrix} 2 & 1 & -3 \end{bmatrix} \end{array}$$

In this method, we will reduce Row<sub>1</sub> by making  $a_{11} = 1$  and the rest of the elements in Column<sub>1</sub> as 0 by using the elementary row transformations

$$R_2 \rightarrow R_2 - 2R_1 \Rightarrow \begin{bmatrix} 1 & -3 & 9 \\ 0 & 7 & -21 \end{bmatrix}$$

Then we shall rewrite the system of equations, using the above row reduced matrix:

$$R_1 \rightarrow x - 3y = 9 \dots\dots(i)$$

And  $R_2 \rightarrow 0x + 7y = -21 \Rightarrow y = -3$

Substituting value of  $y$  in equation (i), we get:

$$x - 3(-3) = 9 \Rightarrow x = 0$$

### Example 45

Use row reduction method to solve the given system of equations:

$$3x + 2y - z = 1$$

$$x + 2y - 2z = 0$$

$$\text{And } 2x + y - 3z = -1$$

**Solution:** Let  $3x + 2y - z = 1 \dots\dots (i)$

$$x + 2y - 2z = 0 \dots\dots(ii)$$

$$\text{And } 2x + y - 3z = -1 \dots\dots (iii)$$

Write the given system of equation in the augmented matrix as

$$\begin{array}{lcl} \text{Equation (i)} & \longrightarrow & \begin{bmatrix} 3 & 2 & -1 & 1 \end{bmatrix} \\ \text{Equation (ii)} & \longrightarrow & \begin{bmatrix} 1 & 2 & -2 & 0 \end{bmatrix} \\ \text{Equation (iii)} & \longrightarrow & \begin{bmatrix} 2 & 1 & -3 & -1 \end{bmatrix} \end{array}$$

We will begin by converting  $a_{11} = 1$  and the rest of the elements in Column<sub>1</sub> as 0 by using the elementary row transformations

$$R_1 \leftrightarrow R_2 \Rightarrow \begin{bmatrix} 1 & 2 & -2 & 0 \\ 3 & 2 & -1 & 1 \\ 2 & 1 & -3 & -1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 3R_1 \Rightarrow \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -4 & 5 & 1 \\ 2 & 1 & -3 & -1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 2R_1 \Rightarrow \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -4 & 5 & 1 \\ 0 & -3 & 1 & -1 \end{bmatrix}$$

$$R_2 \rightarrow \frac{R_2}{-4} \Rightarrow \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & 1 & -\frac{5}{4} & -\frac{1}{4} \\ 0 & -3 & 1 & -1 \end{bmatrix}$$



$$R_2 \rightarrow R_3 + 3R_2 \Rightarrow \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & 1 & \frac{-5}{4} & \frac{-1}{4} \\ 0 & 0 & \frac{-11}{4} & \frac{-7}{4} \end{bmatrix}$$

$$R_3 \rightarrow \frac{-4R_3}{11} \Rightarrow \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & 1 & \frac{-5}{4} & \frac{-1}{4} \\ 0 & 0 & 1 & \frac{7}{11} \end{bmatrix}$$

Now we shall write the above matrix into a system of equations one more time:

$$x + 2y - 2z = 0$$

$$y - \frac{5}{4}z = \frac{-1}{4}$$

$$\text{and } z = \frac{7}{11}$$

$$\Rightarrow y - \frac{5}{4} \times \frac{7}{11} = \frac{-1}{4} \Rightarrow y = \frac{6}{11}$$

$$\text{Also } x + 2 \times \frac{6}{11} - 2 \times \frac{7}{11} = 0 \Rightarrow x = \frac{2}{11}$$

$$\text{Therefore } x = \frac{2}{11}, y = \frac{6}{11} \text{ and } z = \frac{7}{11}$$

#### Example 46

Three shopkeepers A, B and C are using polythene bags, handmade bags and newspaper bags. A uses 20, 30 and 40 number of bags of respective type. B uses 30, 40 and 20 of each respective kind while C uses 40, 20 and 30 of each type. Each shopkeeper spent Rs 250, Rs 220 and Rs 200 on the bags. Find the cost of each carry bag using matrix method.

**Solution:**

| Shopkeeper | Polythene bags | Handmade bags | Newspaper bags | Total cost (in ₹) | Cost per bag |
|------------|----------------|---------------|----------------|-------------------|--------------|
| A          | 20             | 30            | 40             | 250               | x            |
| B          | 30             | 40            | 20             | 220               | y            |
| C          | 40             | 20            | 30             | 200               | z            |

As per the question,

$$20x + 30y + 40z = 250 \Rightarrow 2x + 3y + 4z = 25$$

$$30x + 40y + 20z = 220 \Rightarrow 3x + 4y + 2z = 22$$

$$\text{and, } 40x + 20y + 30z = 200 \Rightarrow 4x + 2y + 3z = 20$$

Writing in matrix form:

$$\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 2 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 25 \\ 22 \\ 20 \end{bmatrix}$$

$$\text{Where } A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

$$\text{As } |A| = 2(12 - 4) - 3(9 - 8) + 4(6 - 16) = 16 - 3 - 40 = -27 \neq 0 \Rightarrow A^{-1} \text{ exists}$$

$$\text{Adj } A = \begin{bmatrix} 8 & -1 & -10 \\ -1 & -10 & 8 \\ -10 & 8 & -1 \end{bmatrix}' = \begin{bmatrix} 8 & -1 & -10 \\ -1 & -10 & 8 \\ -10 & 8 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} (\text{adj } A) = \frac{-1}{27} \begin{bmatrix} 8 & -1 & -10 \\ -1 & -10 & 8 \\ -10 & 8 & -1 \end{bmatrix}$$

$$\text{As } AX = B \text{ then } X = A^{-1} B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{27} \begin{bmatrix} 8 & -1 & -10 \\ -1 & -10 & 8 \\ -10 & 8 & -1 \end{bmatrix} \begin{bmatrix} 25 \\ 22 \\ 20 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{27} \begin{bmatrix} 25 \\ 22 \\ 20 \end{bmatrix} = \begin{bmatrix} \frac{22}{27} \\ \frac{85}{27} \\ \frac{94}{27} \end{bmatrix}$$

Therefore cost of a polythene bag = ₹  $\frac{22}{27}$

cost of a handmade bag = ₹  $\frac{85}{27}$

cost of a newspaper bag = ₹  $\frac{94}{27}$

### Example 47

A school plans to award ₹ 6000 in total to its students to reward for certain values - honesty, regularity and hard work. When three times the award money for hard work is added to the award money given for honesty amounts to ₹ 11000. The award money for honesty and hard work together is double the award money for regularity. Use matrix method to find the prize money for each category of award.

**Solution:** Let the prize money for honesty = ₹ x

Prize money for regularity = ₹ y

And prize money for hard work = ₹ z

As per the question.  $x + y + z = 6000$

$x + 3z = 11000$

and  $x + z = 2y \Rightarrow x - 2y + z = 0$

$$\text{Writing in matrix form: } \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6000 \\ 11000 \\ 0 \end{bmatrix}$$

$$\text{Where } A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{bmatrix}$$

As  $|A| = 1(0 + 6) - 1(1 - 3) + 1(-2 + 0) = 6 + 2 - 2 = 6 \neq 0 \Rightarrow A^{-1}$  exists

$$\text{adj } A = \begin{bmatrix} 6 & 2 & -2 \\ -3 & 0 & 3 \\ 3 & -2 & -1 \end{bmatrix}' = \begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} (\text{adj } A) = \frac{1}{6} \begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}$$

$$\text{As } AX = B \text{ then } X = A^{-1} B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix} \begin{bmatrix} 6000 \\ 11000 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 3000 \\ 12000 \\ 21000 \end{bmatrix} = \begin{bmatrix} 500 \\ 2000 \\ 3500 \end{bmatrix}$$

Therefore, prize money for each prize for honesty = ₹ 500

Prize money for each prize for regularity = ₹ 2000

Prize money for each prize for hard work = ₹ 3500

## 2.8.4 CHECK YOUR PROGRESS

### EXERCISE – D

1. Find the inverse of the given matrices, by using elementary transformations:

i)  $\begin{bmatrix} -5 & -1 \\ 3 & 2 \end{bmatrix}$

ii)  $\begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix}$

iii)  $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$

iv)  $\begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$

2. Find inverse of the given matrices, by using Adjugate (matrix) method:

i.  $\begin{bmatrix} 4 & -1 \\ 3 & 2 \end{bmatrix}$

ii.  $\begin{bmatrix} 2 & -1 & 4 \\ 4 & 0 & 2 \\ 3 & -2 & 7 \end{bmatrix}$

3. Solve the following system of equations by i) Matrix method ii) Row reduction method:

a)  $2x - 3y = -4$ ,  $3x + 5y = 13$

b)  $x + y = 1$ ,  $5x - 7y = 29$

c)  $5x - 4y = 9$ ,  $3x + 7y = -4$

d)  $x - y + 2z = 1$ ,  $2y - 3z = 1$ ,  $3x - 2y + 4z = 2$

e)  $2x - 3y + 5z = 1$ ,  $3x + 2y - 4z = -5$ ,  $x + y - 2z = -3$

4. Solve the following system of equation using Cramer's rule.

i)  $2x - 3y = -4$ ,  $3x + 5y = 13$

ii)  $x + y = 1$ ,  $5x - 7y = 29$

iii)  $5x - 4y = 9$ ,  $3x + 7y = -4$

iv)  $x - 3y = 4$ ,  $3x - 9y = 12$

v)  $-2x + y = 3$ ,  $4x - 2y = 5$

vi)  $x - y + 2z = 1$ ,  $2y - 3z = 1$ ,  $3x - 2y + 4z = 2$

vii)  $2x - 3y + 5z = 1$ ,  $3x + 2y - 4z = -5$ ,  $x + y - 2z = -3$

viii)  $x + y + z = 0$ ,  $-3x + y - 4z = 0$ ,  $-2x + 2y - 3z = 0$

ix)  $2x - y - 3z = 1$ ,  $3x + 2y - 5z = 0$ ,  $5x + y - 8z = 3$

## 2.9 APPLICATION OF MATRICES AND DETERMINANTS

Matrices and determinants are powerful tools in modern mathematics, which have a wide range of application. Sociologists use matrices to study the dominance within a social group or society. Demographers make use of matrices to study survival of mankind, marriage and decent structure. Business mathematics, economist, artificial intelligence coding and networking models are a few examples that function on the concept of matrix and determinant. The study of communication theory and electrical engineering as the network analysis is also aided by the use of matrix representation.

### 2.7.1 Leontief input-output model for two variables

The economy of any country is dependent on many sectors which are interlinked with each other. In this section we try to learn whether the interlinks are viable or not. If the system is viable then try to find the interlink inputs based on the demand in the market. Many sectors are sharing their resources and try to become self-sufficient and independent.

To understand the technique, let's consider two sectors, Automobile Sector (AS) and Electrical Sector (ES). Both are dependent on each other. As part of the resources are used by one industry and remaining resources are shared with other sectors.

Consider the situation based on the given information, that automobile sector uses 20 units of resources for themselves and share 15 units to electrical sector. The electrical sector is using 30 units of resources for themselves and share 12 units with the automobile sector.

As many sectors share their respective resources with each other, In the given situation let the total resources produced by AS be 60 and ES as 45

Let us represent this situation in the following manner:

|                          | AS | ES | — | — | — | Total |
|--------------------------|----|----|---|---|---|-------|
| Resources produced by AS | 20 | 15 | — | — | — | 60    |
| Resources produced by ES | 12 | 30 | — | — | — | 45    |

But make sure that no column additions are possible as the units are different.

|                          | AS    | ES    | — | — | — | Total |
|--------------------------|-------|-------|---|---|---|-------|
| Resources produced by AS | 20/60 | 15/45 | — | — | — | 60    |
| Resources produced by ES | 12/60 | 30/45 | — | — | — | 45    |

Now we try to find per unit worth of output

This representation is called input-output coefficient unit matrix.

$$\text{I/O Coefficient unit Matrix} = \begin{bmatrix} \frac{20}{60} & \frac{15}{45} \\ \frac{12}{60} & \frac{30}{45} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{5} & \frac{2}{3} \end{bmatrix}$$

Now let us introduce one more additional information that every sector is in demand at a particular time, which may change with respect to time.

So the demand matrix is always changing accordingly there is impact on coefficient matrix.

Let us consider the demand as follows

|                          | AS | ES | - | - | — | Total | Demand |
|--------------------------|----|----|---|---|---|-------|--------|
| Resources produced by AS | 20 | 15 | - | - | - | 60    | 70     |
| Resources produced by ES | 12 | 30 | - | - | — | 45    | 50     |

Now we shall convert problem into matrix form

$$AX + D = X \quad \dots\dots\dots(1)$$

where A is an input-output unit matrix(technology matrix).

D is the total demand

X is the new requirement output.

From equation (1)  $D = IX - AX$

$\Rightarrow D = (I - A) X$

Therefore, the requirement output  $X = (I-A)^{-1}D$

Hawkins-Simon Conditions to check for **viability** of an economy are defined as:

- i)  $|I - A|$  must be positive
- ii) Diagonal element of  $I - A$  must be positive.

Let us take an example to test the above stated conditions

Here,  $A = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{5} & \frac{2}{3} \end{bmatrix}$

then  $I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{5} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{5} & \frac{1}{3} \end{bmatrix}$

$\Rightarrow |I-A| = \begin{vmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{5} & \frac{1}{3} \end{vmatrix} = \frac{2}{9} - \frac{1}{15} = \frac{7}{45} > 0$  which verifies the first Hawkins-Simon condition

Also main diagonal of  $I-A$  is positive.

$\Rightarrow$  both of the Hawkins-Simon conditions are satisfied.

Hence required input to fulfil the demand and is viable

The requirement output  $X = (I-A)^{-1}D$

$\Rightarrow X = \frac{1}{|I-A|} \text{adj}(I-A).D$

In the above example,

$$X = \frac{1}{7/45} \begin{bmatrix} 1/3 & 1/3 \\ 1/5 & 2/3 \end{bmatrix} \begin{bmatrix} 70 \\ 50 \end{bmatrix}$$

$$X = \begin{bmatrix} 5400/21 \\ 31950/135 \end{bmatrix}$$

Hawkins-Simon Condition

- $|I-A|$  must be positive
- Diagonal element of  $I-A$  must be positive.

### Example 48

Prepare an input-output table for Transport industry (TI) and Food industry (FI). Food industry produces 50 units. Out of these 20 units consumed by FI and 25 units by TI. Whereas Transport industry produces 40 units and out of these 10 units used by FI and 25 units by TI.

Construct input- output matrix. Check the condition of Hawkins-Simon condition and decide whether system is viable. If so find the input to fulfil the demand 100 of FI and 80 of TI.

**Solution:**

|                    | FI | TI | Total |
|--------------------|----|----|-------|
| Food Industry      | 20 | 25 | 50    |
| Transport Industry | 10 | 25 | 40    |

$$\text{Input-output coefficient matrix (A)} = \begin{bmatrix} 20/50 & 25/40 \\ 10/50 & 25/40 \end{bmatrix} = \begin{bmatrix} 2/5 & 5/8 \\ 1/5 & 5/8 \end{bmatrix}$$

Firstly we will test the viability using Hawkins-Simon condition

$$\text{Get } I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2/5 & 5/8 \\ 1/5 & 5/8 \end{bmatrix} = \begin{bmatrix} 3/5 & -5/8 \\ -1/5 & 3/8 \end{bmatrix}$$

$$\text{Now } |I-A| = \begin{vmatrix} 3/5 & -5/8 \\ -1/5 & 3/8 \end{vmatrix} = \frac{9}{40} - \frac{1}{8} = \frac{9-5}{40} = \frac{1}{10}$$

Clearly diagonal positions of I-A are positive. System is viable.

$$\text{Here demand matrix is given as } = D = \begin{bmatrix} 100 \\ 80 \end{bmatrix}$$

$$\text{So } X = (I-A)^{-1}D$$

$$X = \frac{1}{|I-A|} \text{adj}(I-A).D$$

$$X = \frac{1}{10} \begin{bmatrix} 3/8 & 5/8 \\ 1/5 & 3/5 \end{bmatrix} \begin{bmatrix} 100 \\ 80 \end{bmatrix}$$

$$X = \frac{1}{10} \begin{bmatrix} \frac{300}{8} + \frac{400}{8} \\ \frac{100}{5} + \frac{240}{5} \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 700/8 \\ 340/5 \end{bmatrix} = \begin{bmatrix} 70/8 \\ 34/5 \end{bmatrix}$$

Hence, to fulfil the demand FI must share 70/8 units and TI must share 34/5 units.

## 2.9.2 CHECK YOUR PROGRESS

### EXERCISE – E

1. Solve the following problems using Leontief input-output model.

|          | Sector 1 | Sector 2 | Total |
|----------|----------|----------|-------|
| Sector 1 | 2        | 5        | 10    |
| Sector 2 | 3        | 4        | 20    |

If the system is viable then discuss the situation for new demand 8 and 12 from sector 1 and sector 2 respectively.

2. Solve the following problems using Leontief input-output model.

|                       | FI | AI | Total |
|-----------------------|----|----|-------|
| Food industry         | 20 | 10 | 40    |
| Agricultural industry | 30 | 20 | 60    |

If the system is viable then discuss the situation for new demand 80 and 120 from FI and AI respectively.

3. Solve the following problems using Leontief input-output model.

|          | Sector 1 | Sector 2 | Total |
|----------|----------|----------|-------|
| Sector 1 | 12       | 20       | 40    |
| Sector 2 | 15       | 20       | 30    |

If the system is viable then discuss the situation for new demand 8 and 8 from sector 1 and sector 2 respectively.

4. Solve the following problems using Leontief input-output model.

|          | Sector 1 | Sector 2 | Total |
|----------|----------|----------|-------|
| Sector 1 | 5        | 7        | 30    |
| Sector 2 | 6        | 14       | 21    |

If the system is viable then discuss the situation for new demand 8 and 12 from sector 1 and sector 2 respectively.

## 2.10 UNIT SUMMARY

1. A rectangular array or table of numbers, symbols, expressions or functions, when arranged in rows and columns, is known as a *matrix* (plural *matrices*). Each member of this arrangement is called an *element* of the matrix.
2. A matrix is expressed by using capital English alphabet, say  $A = [a_{ij}]$  where  $a_{ij}$  is the element at the  $i$ th row and  $j$ th column of the matrix and  $1 \leq i \leq m, 1 \leq j \leq n, \forall i, j \in N$

3. For a matrix  $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$  having 'm' number of rows and 'n' number of columns, the expression  $m \times n$  is called the *order* of the given matrix A where  $1 \leq i \leq m, 1 \leq j \leq n, \forall i, j \in N$

The elements  $a_{11}, a_{22}, a_{33}, \dots$  where  $a_{ij}, \forall i = j$  are called elements of the *diagonal* of the matrix

And the elements where  $a_{ij}, \forall i \neq j$  are called elements of the *non-diagonal* of the matrix

4. A matrix in which number of rows is not equal to number of columns is called a *Rectangular matrix*
5. A matrix in which the number of rows and columns are equal is called a *Square matrix*
6. A matrix having exactly one row is called a *Row matrix*.
7. A matrix having exactly one column is called a *Column matrix*.
8. A square matrix in which all the non-diagonal entries are zero, i.e.  $a_{ij} = 0, \forall i \neq j$  is called a *Diagonal matrix*
9. A diagonal matrix having same diagonal elements, i.e.  $a_{ij} = k, \forall i = j$  where  $k \neq 0$  is called a *Scalar matrix*.



10. A scalar matrix in which all the diagonal entries are equal to 1,  
i.e.  $a_{ij} = 1, \forall i = j$  is called an *Identity matrix (Unit matrix)*, denoted by English alphabet I
11. A matrix with each of its elements as zero,  
i.e.  $a_{ij} = 0, \forall 1 \leq i \leq m, 1 \leq j \leq n, \forall i, j \in N$  is called a *zero matrix*
12. Two matrices  $A = [a_{ij}]$  and  $B = [b_{ij}]$  having same order  $m \times n$  are called *Equal matrices* when each element of A is equal to the corresponding element of B,  
i.e.  $a_{ij} = b_{ij} \forall 1 \leq i \leq m \text{ and } 1 \leq j \leq n$
13. For a matrix  $A = [a_{ij}]$  of order  $m \times n$  and  $k$  is a scalar quantity, then  $kA$  is another matrix obtained by multiplying each element of A by the scalar quantity  $k$ ,  
i.e.  $kA = k [a_{ij}] = [ka_{ij}], \forall 1 \leq i \leq m \text{ and } 1 \leq j \leq n$
14. If  $A = [a_{ij}]$  and  $B = [b_{ij}]$  are two matrices of the same order, say  $m \times n$ , and  $k$  and  $p$  are scalars, then
  - (i)  $k(A + B) = kA + kB$
  - (ii)  $(k + p)A = kA + pA$
  - (iii)  $k(A + B) = kA + kB$
15. For a non-zero matrix A, of order  $m \times n$ , a matrix B of same order is called *Negative matrix* of matrix A such that  $A + B = O$ , where O is the zero matrix of the same order. We denote negative matrix A as  $-A$
16. For two matrices  $A = [a_{ij}]$  and  $B = [b_{ij}]$ , of same order  $m \times n$ , the sum of two matrices A and B is defined as a matrix  $S = A + B = [s_{ij}]$  of order  $m \times n$  such that  
 $s_{ij} = a_{ij} + b_{ij} \forall 1 \leq i \leq m \text{ and } 1 \leq j \leq n$
17. Addition of two or more matrices is possible only when the given matrices are of same order. The order of resultant matrix is also same  
 $\Rightarrow A_{m \times n} + B_{m \times n} = C_{m \times n}$   
 Matrix addition is commutative  $\Rightarrow A_{m \times n} + B_{m \times n} = B_{m \times n} + A_{m \times n}$   
 Matrix addition is associative  $\Rightarrow A_{m \times n} + (B_{m \times n} + C_{m \times n}) = (A_{m \times n} + B_{m \times n}) + C_{m \times n}$   
 Zero matrix is the additive identity  $\Rightarrow A_{m \times n} + O_{m \times n} = A_{m \times n} = O_{m \times n} + A_{m \times n}$   
 Negative of a matrix is the additive inverse in matrix addition  $\Rightarrow A_{m \times n} + (-A_{m \times n}) = O_{m \times n}$
18. For two matrices  $A = [a_{ij}]$  and  $B = [b_{ij}]$ , of same order  $m \times n$ , the difference of two matrices A and B is defined as a matrix  $D = A - B = [d_{ij}]$  of order  $m \times n$  such that  $d_{ij} = a_{ij} - b_{ij} \forall 1 \leq i \leq m \text{ and } 1 \leq j \leq n$
19. For two matrices  $A = [a_{ij}]$  of order  $m \times n$  and  $B = [b_{jk}]$  of order  $n \times p$ , the multiplication of the matrices AB is defined as a matrix  $P = AB = [p_{ik}]$  of order  $m \times p$  such that for finding the  $p_{ik}$  we multiple  $i$ th row of first matrix A with the  $k$ th column of second matrix B and calculate the sum of these products  
i.e.  $p_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + a_{i3}b_{3k} + a_{i4}b_{4k} + \dots + a_{in}b_{nk} \forall 1 \leq i \leq m \text{ and } 1 \leq j \leq n$

20. i. The multiplication of matrices is associative i.e. for any three matrices A, B and C,  $(AB)C = A(BC)$ , whenever order of multiplication is defined on both sides
- ii. Distributive property of multiplication holds true for multiplication of matrices. i.e. for three matrices A, B and C,  $A(B+C) = AB + AC$
- iii.  $A(B - C) = AB - AC$ , whenever order of multiplication is defined on both sides.
21. For a given square matrix A of order  $m \times m$ , there exists a multiplicative identity matrix  $I$ , of same order such that  $IA = AI = A$ .
22. For a matrix  $A = [a_{ij}]$  of order  $m \times n$ , the matrix obtained by interchanging the rows and columns of the matrix is called the transpose of matrix A
23. For a matrix A of order  $m \times n$ , the order of transpose matrix A is  $n \times m$
24. For given matrices A and B:
  - i.  $(A')' = A$
  - ii.  $(kA)' = kA'$  (where k is any constant)
  - iii.  $(A + B)' = A' + B'$
  - iv.  $(AB)' = B'A'$
24. For a given square matrix  $A = [a_{ij}]$ , if  $A' = A$ ,  $\forall 1 \leq i \leq m \text{ and } 1 \leq j \leq n$ , then the matrix A is called a symmetric matrix
25. For a given square matrix  $A = [a_{ij}]$ , if  $A' = -A$ ,  $\forall 1 \leq i \leq m \text{ and } 1 \leq j \leq n$ , then the matrix A is called a skew-symmetric matrix
26. For a square matrix A having real values as elements,
  - i.  $A + A'$  is a symmetric matrix
  - ii.  $A - A'$  is a skew symmetric matrix
27. Let X be a set of square matrices and R be the set of numbers (real or complex) such that a function f is defined as  $f : M \rightarrow R$  by  $f(A) = k$ , where  $A \in X$  and  $k \in R$ , then  $f(A)$  is called the determinant of A
28. In a given determinant, minor of an element  $a_{ij}$  is the determinant obtained by deleting its  $i$ th row and  $j$ th column in which element  $a_{ij}$  lies, and is denoted by  $M_{ij}$ .
29. In a square matrix, cofactor of an element  $a_{ij}$ , denoted by  $A_{ij}$  or  $C_{ij}$  is defined by  $A_{ij} = (-1)^{i+j} M_{ij}$ , where  $M_{ij}$  is the minor of  $a_{ij}$
30. The transpose of cofactors matrix of a square matrix  $A = [a_{ij}]$  is called Adjoint matrix and is denoted by  $\text{adj } A$ .
31. For a given square matrix A, of order n,
  - i.  $A(\text{adj } A) = (\text{adj } A)A = |A| I$ , where  $I$  is the identity matrix of order n
  - ii.  $|\text{adj}(A)| = |A|^{n-1}$
32. If the determinant of any matrix is zero then that matrix is called Singular Matrix and if  $|A| \neq 0$  then A is called Non-singular matrix.
33. Let  $A(x_1, y_1)$ ,  $B(x_2, y_2)$  and  $C(x_3, y_3)$  be the vertices of the triangle ABC, then

$$\text{Area } \Delta ABC = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

34. For a given square matrix A of order n, if there exists another square matrix B of the same order n, such that  $AB = BA = I$ , then A is said to be invertible and B is called the inverse matrix of A and denoted by  $A^{-1}$
- Inverse of a matrix, if it exists, is unique
  - For two invertible matrices of same order, say A and B, then  $(AB)^{-1} = B^{-1} A^{-1}$
  - For an invertible matrix A,  $(A^{-1})^{-1} = A$
  - For an invertible matrix A,  $(A^T)^{-1} = (A^{-1})^T$
35. A system of equations can be solved by using any of the following methods:
- inverse of coefficient method
  - Cramer's method
  - Row reduction method
36. Hawkins-Simon Conditions to check for **viability** of an economy are defined as:
- $|I - A|$  must be positive
  - Diagonal elements of  $(I - A)$  must be positive

## 2.11. ANSWERS TO THE EXERCISES

### EXERCISE – A

- Row,  $1 \times 2$
  - Column,  $3 \times 1$
  - Rectangle,  $3 \times 2$
  - Row  $1 \times 3$
  - Square  $2 \times 2$
  - Square  $3 \times 3$
  - Zero  $2 \times 2$
  - Zero  $2 \times 3$
  - Identity  $3 \times 3$
  - Scalar  $3 \times 3$
- 4
  - $0 + 2$
  - 4
- $$\begin{bmatrix} \frac{9}{2} & \frac{25}{2} & \frac{49}{2} \\ 8 & 18 & 32 \end{bmatrix}$$
- $$\begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \end{bmatrix}$$
- 81
- 4
- $a = 14, b = -7, c = 7, d = 1$

## EXERCISE – B

1. Order of the matrix

| A            | B            | $A \pm B$    | AB           |
|--------------|--------------|--------------|--------------|
| $2 \times 2$ | $2 \times 2$ | $2 \times 2$ | $2 \times 2$ |
| $2 \times 3$ | $3 \times 2$ | Not possible | $2 \times 2$ |
| $3 \times 4$ | $4 \times 1$ | Not possible | $3 \times 1$ |
| $3 \times 3$ | $3 \times 3$ | $3 \times 3$ | $3 \times 3$ |
| $2 \times 3$ | $2 \times 3$ | $2 \times 3$ | Not possible |
| $1 \times 3$ | $3 \times 2$ | Not possible | $1 \times 2$ |

5. i)  $\begin{bmatrix} 35 & -20 \\ -28 & 27 \end{bmatrix}$

ii)  $[-4]$

iii)  $\begin{bmatrix} 8 & -3 & 5 \\ -2 & -3 & -6 \end{bmatrix}$

iv)  $p=1, q=2, r=3, s=4$

v)  $\begin{bmatrix} 4 & -11 \\ -5 & -1 \end{bmatrix}$

6.  $\begin{bmatrix} -5 & -1 \\ -1 & 5/2 \\ -19/4 & -4 \end{bmatrix}$

7. i.  $\begin{bmatrix} -5 & -8 & 12 \\ -8 & 0 & 20 \\ -6 & 5 & -11 \end{bmatrix}$

ii.  $\begin{bmatrix} -5 & -8 & 12 \\ -8 & 0 & 20 \\ -6 & 5 & -11 \end{bmatrix}$

iii.  $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

iv.  $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

v.  $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

9. In the month of March  $P = \begin{bmatrix} 250 & 400 \\ 230 & 425 \end{bmatrix} \begin{matrix} \text{bookseller A} \\ \text{bookseller B} \end{matrix}$

In the month of April  $Q = \begin{bmatrix} 550 & 300 \\ 270 & 450 \end{bmatrix} \begin{matrix} \text{bookseller A} \\ \text{bookseller B} \end{matrix}$

$$\begin{aligned} \text{Total Sell } P+Q &= \begin{bmatrix} 250 & 400 \\ 230 & 425 \end{bmatrix} + \begin{bmatrix} 550 & 300 \\ 270 & 450 \end{bmatrix} \\ &= \begin{bmatrix} 800 & 700 \\ 500 & 875 \end{bmatrix} \begin{matrix} \text{bookseller A} \\ \text{bookseller B} \end{matrix} \end{aligned}$$

10. Sell matrix

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$$S = \begin{bmatrix} 5 & 7 \\ 6 & 4 \end{bmatrix} \begin{matrix} \text{bookseller P} \\ \text{bookseller Q} \end{matrix}$$

Cost Matrix

$$C = \begin{bmatrix} 12 \\ 27 \end{bmatrix}$$

Amount Matrix

$$A = S C$$

$$\begin{aligned} A &= \begin{bmatrix} 5 & 7 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} 12 \\ 27 \end{bmatrix} \\ &= \begin{bmatrix} 60 + 189 \\ 72 + 108 \end{bmatrix} \\ &= \begin{bmatrix} 249 \\ 180 \end{bmatrix} \end{aligned}$$

Bookseller P generates the amount Rs 249/- and Q generates the amount Rs 180/-

### EXERCISE – C

1. i) 0, ii) -53, iii) 25, iv) 0, v) 18
2. 15 sq. units
3. -5

7. i.  $\begin{bmatrix} 5 & -1 \\ 3 & 2 \end{bmatrix}$

ii.  $\begin{bmatrix} 51 & -11 \\ 0 & -52 \end{bmatrix}$

iii.  $\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -2 \\ 2 & -1 & 5 \end{bmatrix}$

### EXERCISE – D

1. i)  $\frac{1}{-7} \begin{bmatrix} 2 & 1 \\ -3 & -5 \end{bmatrix}$

ii)  $\begin{bmatrix} -2 & -1 \\ 5 & 3 \end{bmatrix}$

iii)  $\frac{1}{10} \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$

iv)  $:\frac{1}{5} \begin{bmatrix} 2 & 1 & -5 \\ 1 & -2 & 5 \\ -3 & 1 & 5 \end{bmatrix}$

2. i)  $\frac{1}{11} \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}$

ii)  $\begin{bmatrix} -2 & 1/2 & 1 \\ 11 & -1 & -6 \\ 4 & -1/2 & -2 \end{bmatrix}$

3. a)  $x = 1, y = 2$   
 c)  $x = 1, y = -1$   
 e)  $x = 1, y = 2, z = 3$

- b)  $x = 3, y = -2$   
 d)  $x = 0, y = 5, z = 3$

- 4.
- |                                  |                                 |
|----------------------------------|---------------------------------|
| i) $x = 1$ , $y = 2$             | ii) $x = 3$ , $y = -2$          |
| iii) $x = 1$ , $y = -1$ )        | iv) Infinite many solution      |
| v) No solution                   | vi) $x = 0$ , $y = 5$ , $z = 3$ |
| vii) $x = 1$ , $y = 2$ , $z = 3$ | viii) Infinite many solution    |
| ix) No solution                  |                                 |

## EXERCISE – E

1. Coefficient unit matrix =  $\begin{bmatrix} 2/10 & 5/20 \\ 3/10 & 4/20 \end{bmatrix} = \begin{bmatrix} 1/5 & 1/4 \\ 3/10 & 1/5 \end{bmatrix}$

Get  $I-A = \begin{bmatrix} 4/5 & -1/4 \\ -3/10 & 4/5 \end{bmatrix},$

Check conditions  $\det(I-A) = 16/25 - 3/40 = 109/200 > 0$ , main diagonal entries positive

$$\text{So } X = (I-A)^{-1}D = \frac{200}{109} \begin{bmatrix} 4/5 & 1/4 \\ 3/10 & 4/5 \end{bmatrix} \begin{bmatrix} 8 \\ 12 \end{bmatrix}$$

2. Coefficient unit matrix =  $\begin{bmatrix} 20/40 & 10/60 \\ 30/60 & 20/60 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/6 \\ 1/2 & 1/3 \end{bmatrix}$

Get  $I-A = \begin{bmatrix} 1/2 & -5/6 \\ -1/2 & 2/3 \end{bmatrix},$

Check conditions  $\det(I-A) = 1/6 - 5/12 = -3/12 < 0$ ,

First condition of Hawkins-Simon is not true. So condition is **not viable**

3. Coefficient unit matrix =  $\begin{bmatrix} 12/40 & 20/30 \\ 15/40 & 20/30 \end{bmatrix} = \begin{bmatrix} 3/10 & 2/3 \\ 3/8 & 2/3 \end{bmatrix}$

Get  $I-A = \begin{bmatrix} 7/10 & -2/3 \\ -3/8 & 1/3 \end{bmatrix},$

Check conditions  $\det(I-A) = 7/30 - 1/4 = (14-15)/60 < 0$ ,

And main diagonal entries positive.

First condition of Hawkins-Simon is not true. So condition is **not viable**.

4. Coefficient unit matrix =  $\begin{bmatrix} 5/30 & 7/21 \\ 6/30 & 14/21 \end{bmatrix} = \begin{bmatrix} 1/6 & 1/3 \\ 1/5 & 2/3 \end{bmatrix}$

Get  $I-A = \begin{bmatrix} 5/6 & -1/3 \\ -1/5 & 1/3 \end{bmatrix}$ ,

Check conditions  $\det(I-A) = 5/18 - 1/15 = (25-6)/90 = 19/90 > 0$ , main diagonal entries positive

$$\text{So } X = (I-A)^{-1}D = \frac{90}{19} \begin{bmatrix} 2/3 & -1/3 \\ -1/5 & 1/6 \end{bmatrix} \begin{bmatrix} 8 \\ 12 \end{bmatrix}$$

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